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A transient solution on potential equation of diffusion type with time-varying coefficients

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Abstract. Research on hydrothermal sulphide ore of marine mineral resources exploration is hot recently. According to the general solution of first order differential equation, Laplace transform is used to transform the linear homogeneous differential equation of second order variable coefficient into first order differential equation of complex domain. The general solution of the first-order differential equations in complex domain is similarly obtained without regard for the difference between real domains and complex domains. Further more, the general solutions are deduced to the second order linear differential equations with variable coefficients. Then it is used to solve the thermoelectric coupling electric field model in oceanic strike exploration; A closed analytical solution with integral form of elementary functions is obtained to the simplified diffusion electric field potential model. To get an explicit solution, with the help of MAPLE software, the general solution of the linear combination of Kummer functions is obtained. This research has theoretical significance for the engineering implementation of the frequency domain induced dipole drag system.

Keywords: Diffusion Electric Potential Equation; Second Order Linear Homogeneous Differential Equation with Variable Coefficients; Laplace Transform ; Kummer Function;

1. Introduction

The transient electromagnetic method and induced polarization method are studied for the exploration of hydrothermal sulfide ore in the domestic. Application of frequency domain induced electrical prospecting, the experimental study showed that can effectively find the metal sulfide of the IP anomalies [1]. Low frequency electromagnetic waves move in seawater according to the diffusion-heat conduction equation. Seawater electromagnetic parameters include permeability μ, conductivity σ and permittivity ε, which are related to the salinity, temperature and density of seawater. Seawater is a non-ferromagnetic substance with a conductivity of $\sigma = 4.54 - 4.81(\Omega \bullet m)^{-1}$ at a temperature of 17°C. The conductivity of submarine hydrothermal sulfide ore is $\sigma = 0.800 - 10.000(\Omega \bullet m)^{-1}$, and the specific value is related to mineral type, metal content, temperature and void density. In this paper, the general solution of the electric field equation of the ocean excitation polarization method is studied when the conductivity σ varies with the temperature $T(t)$.

2. Potential Equation

The Maxwell equation of electromagnetic field of the induced polarization method in time domain is as (1)~(7):

$$\nabla \times h = j + \frac{\partial d}{\partial t}. \tag{1}$$

$$\nabla \times \hat{e} = -\frac{\partial b}{\partial t}. \tag{2}$$

$$\nabla \cdot b = 0. \tag{3}$$

$$\nabla \cdot d = \rho. \tag{4}$$

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The $h, \hat{e}, b, d, j, \rho$ in the formula are magnetic field strength, electric field strength, magnetic induction intensity, potential shift, current density, charge density and t is time coefficient.

The physical property equation is:

$$j = \sigma(t) \cdot \hat{e} . \tag{5}$$

$$d = \varepsilon \cdot \hat{e} . \tag{6}$$

$$b = \mu \cdot h . \tag{7}$$

The σ, ε, μ in the form are conductivity, dielectric constant, permeability.

Considering the variation of conductivity σ with temperature $T(t)$, which affects the measurement of ocean electric field, the heat transfer problem has a solution of $T = T_0 \cdot e^{-at}$, so set up:

$$\sigma = \sigma_0 - A_0 T = \sigma_0 - A e^{-at} . \tag{8}$$

σ_0, A_0, a are constants. Substituting (3) ~ (7) into (1) and (2), can get:

$$\nabla^2 h - \mu\varepsilon \frac{\partial^2 h}{\partial t^2} - \mu\sigma \frac{\partial h}{\partial t} = 0 . \tag{9}$$

$$\nabla^2 \hat{e} - \mu\varepsilon \frac{\partial^2 \hat{e}}{\partial t^2} - \mu\sigma \frac{\partial \hat{e}}{\partial t} = 0 . \tag{10}$$

The above is diffusion heat conduction differential equation. When $\omega \le 10^5 Hz$ is $\mu\varepsilon\omega^2 << \mu\sigma\omega$, the conduction current plays an important role in $\nabla^2 \hat{e} - \mu\sigma \frac{\partial \hat{e}}{\partial t} = 0$, which is the diffusion equation. Steady state $\nabla^2 \hat{e} = 0$ this is the Poisson equation. From (8), (11) is obtained:

$$\nabla^2 \hat{e} - \mu\varepsilon \frac{\partial^2 \hat{e}}{\partial t^2} - \mu\left(\sigma_0 - A e^{-at}\right) \frac{\partial \hat{e}}{\partial t} = 0. \tag{11}$$

Due to symmetry, the polarization field only z component, application of $e^t = 1 + t + \sum_{n=2}^{\infty} \frac{t^n}{n!}$, $t = O(1)$ series expression, neglecting the higher-order terms, there are:

$$\nabla^2 \hat{e} - \mu\varepsilon \frac{\partial^2 \hat{e}}{\partial t^2} - \mu\left(\sigma_0 - A + Aat\right) \frac{\partial \hat{e}}{\partial t} = 0. \tag{12}$$

By using the Semyon Quinn Love potential, the potential equation of the polarization field is rewritten as follows:

$$u_{zz} - p u_{tt} - \left(q + wt\right) u_t = 0. \tag{13}$$

The $u = u(z,t)$, $p = \mu\varepsilon \ge 0$, $q = \mu\left(\sigma_0 - A\right)$, $w = \mu A a$ are constants. In [1], the single-parameter transformation group method is applied to prove that there is no similar solution to the equation with invariant form. Using the method of separation of variables and set $u = u(z,t) = \varphi(z)\Psi(t)$, it can be concluded that:

$$\varphi_{zz}\Psi(t) - p\varphi(z)\Psi_{tt} - (q + wt)\varphi(z)\Psi_t = 0. \tag{14}$$

$$\frac{\varphi''}{\varphi(z)} = \frac{p\Psi'' + (q + wt)\Psi'}{\Psi(t)} = C. \tag{15}$$

C is a constants. $\varphi'' = C\varphi(z)$ can be obtained by $\frac{\varphi''}{\varphi(z)} = C$, and then have:

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$$\varphi(z)=g_1\cos(\sqrt{|C|}z)+g_2\sin(\sqrt{|C|}z). \tag{16}$$

g_1, g_2 are constants. According to the formula $\frac{p\Psi''+(q+wt)\Psi'}{\Psi(t)}=C$,we get:

$$p\Psi''+(q+wt)\Psi'-C\Psi(t)=0. \tag{17}$$

3. The Laplace Transform Solutions of Potential Equations

The second-order linear homogeneous differential equation with variable coefficients encountered in ocean IP exploration is:

$$\psi''(t)+(\alpha+\beta t)\psi'(t)-\gamma\psi(t)=0. \tag{18}$$

$\alpha,\beta>0,\gamma>0$ are constants.

The closed-form solution of second-order linear homogeneous differential equation with variable coefficients can be obtained by: ① Variable substitution method. The original equation is transformed into a second-order homogeneous differential equation with constant coefficients. ② Depression of order method. The original equation is transformed into an total differential equation, and the first-order equation is solved successively. ③ The original equation is solved by Riccati equation [5~8]. ④ The original equation is solved as a system of first-order equations. Try method three to solve, set $m(t)=\alpha+\beta t,$ we get:

$$\psi(t)=\phi(t)e^{-\frac{1}{2}\int m(t)dt}. \tag{19}$$

$$\psi'(t)=\left[\phi'(t)-\frac{1}{2}m(t)\phi(t)\right]e^{-\frac{1}{2}\int m(t)dt}. \tag{20}$$

$$\psi''(t)=\left[\phi''-m\phi'-\frac{1}{2}m'\phi+\frac{1}{4}m^2\phi\right]e^{-\frac{1}{2}\int m(t)dt}. \tag{21}$$

Where $\phi(t)$ is an unknown function of t .Substituting Equation 8, 9 and 10 into Equation 11, we can get:

$$\phi''(t)-\left[\left(\frac{1}{2}\beta t+\frac{1}{2}\alpha\right)^2+\left(\frac{1}{2}\beta+\gamma\right)\right]\phi(t)=0. \tag{22}$$

$\frac{\phi''(t)}{\phi(t)}=(\ln\phi)''+\left[(\ln\phi)'\right]^2$, take $\Phi(t)=\left[\ln\phi(t)\right]'$, then have :

$$\Phi'=-\Phi^2+\left[\left(\frac{1}{2}\beta t+\frac{1}{2}\alpha\right)^2+\left(\frac{1}{2}\beta+\gamma\right)\right]. \tag{23}$$

Equation (23) is a form of the Riccati equation for $\Phi(t)$.

Proposition 1: Here are some formulas:

$$\frac{dy}{dx}=\xi x^\lambda+\nu y^2. \tag{24}$$

$$\lambda=-\frac{4k}{2k\pm1}. \qquad k=0,\pm1,\pm2,... \tag{25}$$

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ν, ξ, λ are constants. Equation (24) is a special Riccati equation and is integrable in finite form of elementary function. If (25) is true, then it can be solved by integrating elementary functions.

Obviously, it is impossible to find the elementary solution of Riccati equation (23). Furthermore, the analytic solution of the potential equation (13) is also unavailable.

The following is a general solution of Riccati equation (23) in the form of special functions with the help of MAPLE, as shown in Fig. 1:

$hypergeom(\cdot)$ is confluent hypergeometric function. $-C1$ is a constant, it is determined by corresponding definite solution conditions.

Since the Laplace transform is often used to solve linear differential equations with constant coefficients, and can also solve linear differential equations with variable coefficients and linear partial differential equations, the Laplace transform method can be used to solve the above problems [11~13]. The Riccati equation of (23) is a first-order differential equation, but it is not linear and does not satisfy the superposition principle. Therefore, it is not suitable for solving Laplace transform. Equation (18) is a second-order linear homogeneous differential equation with variable coefficients can be solved by Laplace transform. It can take:

$$\begin{cases} L[\psi(t)] = \Phi(s). \\ L[\psi'(t)] = s\Phi(s) - \psi(0). \\ L[t\psi'(t)] = -\dfrac{d}{ds}\left[s\Phi(s) - \psi(0)\right] = -\Phi(s) - s\Phi'(s). \\ L[\psi''(t)] = s^2\Phi(s) - s\psi(0) - \psi'(0). \end{cases} \tag{26}$$

That (26) is substituted into (18) can be obtained:

$$\beta s\Phi'(s) + \left(s^2 + \alpha s - \beta - \gamma\right)\Phi(s) - \left(s + \alpha\right)\psi(0) - \psi'(0) = 0. \tag{27}$$

Proposition 2:

$$y'(x) + f_1(x)y = f_2(x). \tag{28}$$

$$y(x) = e^{-\int f_1(x)dx}\left\{\int_{x_0}^{x} f_2(x)e^{\int f_1(x)dx}dx + const\right\}. \tag{29}$$

Equation (28) is a first-order differential equation, and its general solution is shown in (29) ,where $const$ is a constant.

Without considering the difference between the real field and complex field of the function, the general solution of (27) in complex field can be written as follows:

$$\Phi(s) = e^{-\int\frac{s^2+\alpha s-\beta-\gamma}{-\beta s}ds}\left\{\int_{s_0}^{s}\frac{\left(s+\alpha\right)\psi(0)+\psi'(0)}{-\beta s}e^{\int\frac{s^2+\alpha s-\beta-\gamma}{-\beta s}ds}ds + ct\right\}. \tag{30}$$

where ct is a constant.

The general solution of (18) is:

$$\begin{aligned} \psi(t) &= L^{-1}\left[\Phi(s)\right] \\ &= \frac{1}{2\pi j}\int_{\delta-j\infty}^{\delta+j\infty}\Phi(s)e^{st}ds \\ &= \frac{1}{2\pi j}\int_{\delta-j\infty}^{\delta+j\infty}(c_1 s^{-1-\frac{\gamma}{\beta}}e^{\frac{s^2+2\alpha s}{-2\beta}}\{\int_{s0}^{s}\frac{\left(s+\alpha\right)\psi(0)+\psi'(0)}{-\beta}s^{\frac{\gamma}{\beta}} \\ &\qquad \times e^{\frac{s^2+2\alpha s}{-2\beta}}ds + ct\})e^{st}ds. \end{aligned} \tag{31}$$

c_1, ct are constants. and j is imaginary unit. $s = \delta + j\omega$ is complex variable. This is the closed solution of the second order variable coefficient linear homogeneous differential equation (18).

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Applying the above conclusions, set: $\alpha = q/p, \beta = w/p, \gamma = -C/p$ .Substituting the above parameters into (31) and combining with (16), the solution of (13) can be obtained as follows:

$$
\begin{aligned}
u &= u(z,t) \\
&= \varphi(z)\Psi(t) \\
&= [g_1 \cos(\sqrt{|C|}z) + g_2 \sin(\sqrt{|C|}z)] \\
&\quad \times \{\frac{1}{2\pi j}\int_{\delta-j\infty}^{\delta+j\infty}(c_1 s^{-1+\frac{C}{w}} e^{-\frac{ps^2+2qs}{2w}} \{\int_{s0}^{s}\frac{(ps+q)\psi(0)+\psi'(0)}{w} s^{-\frac{C}{w}} \\
&\qquad \times e^{\frac{ps^2+2qs}{2w}} ds + ct\})e^{st} ds\}.
\end{aligned}
\tag{32}
$$

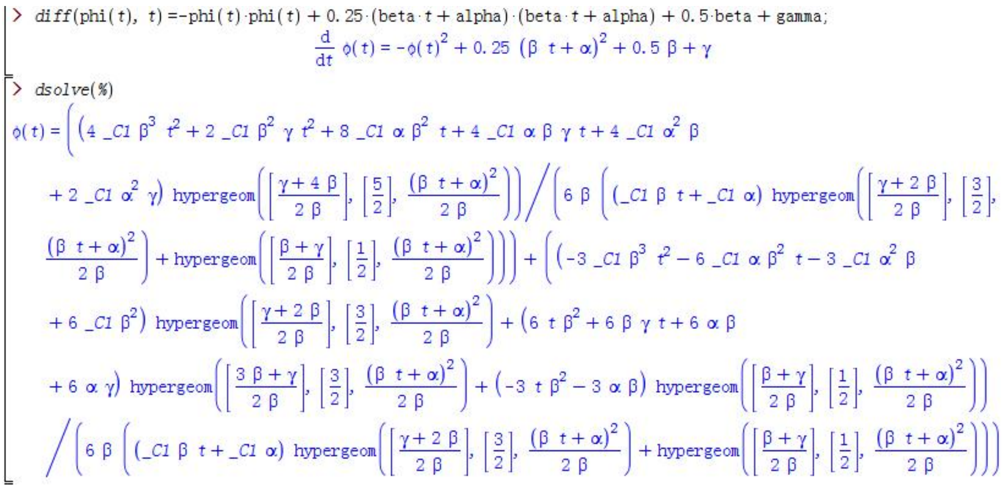

Figure 1. The solution of Riccati equation (23)

Equation (32) is the expression of the solution of (13) in the finite form of elementary function, which cannot clearly show the characteristics of the solution.

4. Special functional Solutions of Potential Equations

The explicit analytical solution cannot be found by applying elementary functions. The following is to seek the general solution of the potential equation (13) in the form of special functions with the help of mathematical software MAPLE [14,15].According to (18), $\psi(t)$ is shown in Fig 2.

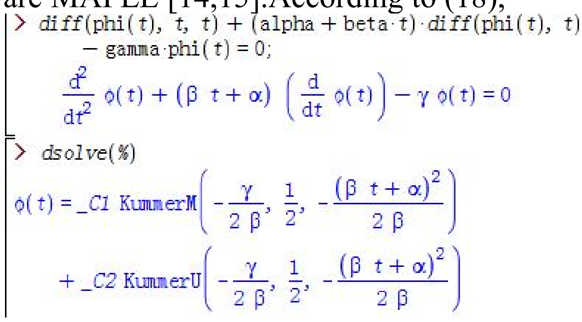

Figure 2. The solution to equation $\psi''(t) + (\alpha + \beta t)\psi'(t) - \gamma\psi(t) = 0$

$KummerM(\cdot)$ is the first kind of confluence hypergeometric function, $KummerU(\cdot)$ is the second kind of confluence hypergeometric function. $\_C1, \_C2$ are constants.

Combining with (16), the solution of (13) can be obtained as follows:

$$
\begin{aligned}
u &= u(z,t) \\
&= \varphi(z)\Psi(t) \\
&= [g_1 \cos(\sqrt{|C|}z) + g_2 \sin(\sqrt{|C|}z)] \\
&\quad \times \{\_C1 \times KummerM(-\frac{\gamma}{2\beta}, \frac{1}{2}, -\frac{(\alpha+\beta t)^2}{2\beta}) \\
&\quad + \_C2 \times KummerU(-\frac{\gamma}{2\beta}, \frac{1}{2}, -\frac{(\alpha+\beta t)^2}{2\beta})\}.
\end{aligned} \tag{33}
$$

5. Conclusion

In this paper, according to the general solution of first order differential equation, Laplace transform is used to transform the linear homogeneous differential equation of second order variable coefficient into first order differential equation of complex domain. The general solution of the first-order differential equations in complex domain is similarly obtained without regard for the difference between real domains and complex domains. Further more, the general solutions are deduced to the second order linear differential equations with variable coefficients. The above conclusions are used to solve the transient solution of electric field model in ocean IP exploration.

In the finite form of elementary function, the transient solution of diffusion electric field potential equation is an integral equation related to complex domain. Although the form is closed, it cannot clearly show the properties of the solution. The author uses MAPLE to find the general solution of the diffusion electric field potential equation in special function form, and gives the closed expression of the transient solution. This also verifies the conclusion in [1] : using the single parameter transformation group method, it is proved that there is no similar solution of the diffusion potential equation with unchanged form. At the same time, it is verified that Riccati equation in the form of (26) has no elementary solution, and a form of its special function solution is given.

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