Study on the Similarity between Nonnegative Irreducible Matrices and Positive Matrices

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Abstract. The main objective of this study is to explore the problem of similarity between nonnegative irreducible matrices and positive matrices. By leveraging established lemmas and drawing insights from pertinent literature, this article provides a rigorous proof: When n is greater than 8, for any $n \times n$ nonnegative irreducible matrix A, if it contains $n + 3$ zero elements and possesses mutually distinct diagonal elements, then A is similar to a positive matrix.

Keywords: nonnegative irreducible matrices, positive matrices, similarity.

1. Introduction

Nonnegative Irreducible Matrices and Positive Matrices hold immense importance in the realm of matrix theory, capturing essential characteristics and properties that have sparked considerable interest among researchers worldwide. Investigating the similarity between these two matrix classes has become a prominent research topic, garnering attention from numerous scholars[1]. Guang-Jing Song and Michael K. Ng [2] have introduced a novel algorithm for computing nonnegative low-rank matrix (NLRM) approximations of nonnegative matrices. Scholars have been investigating the conditions under which a nonnegative matrix and a positive matrix share the same eigenvalues, i.e., are co-spectral. Notably, any nonnegative matrix that is similar to a positive matrix is automatically co-spectral with it. Similarity plays a crucial role as an equivalence relation for matrices, and exploring nonnegative matrices similar to positive matrices is a key approach for addressing co-spectral problems. Additionally, in certain scenarios, it is desirable to assess whether a nonnegative matrix can be similar to a positive matrix without explicit knowledge of its exact element values, merely based on the zero-nonzero patterns of the nonnegative matrix.Scholars have noted that this problem is closely related to the non-negative inverse eigenvalue problem (NIEP) [3]. The nonnegative inverse eigenvalue problem is to characterize the set of n complex numbers $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that σ is the spectrum of some nonnegative matrix. If σ is the spectrum of a nonnegative matrix A, we call σ a realizable spectrum and A a matrix realizing the spectrum σ [4].

In general, the study of certain properties of a class of matrices often begins with investigating low-order matrices, exploring their characteristics, and gradually transitioning to higher-order cases. However, in many situations, the properties of some low-order matrices do not straightforwardly extend to higher-order cases. Therefore, for certain properties of high-order matrices, it is necessary to develop new methods for exploration and research.

To make a nonnegative matrix similar to a positive matrix, the matrix needs to satisfy certain specific conditions. Based on the definition of irreducible matrices and the Perron-Frobenius theorem, we only need to investigate the problem of whether nonnegative irreducible matrices can be similar to positive matrices.In recent years, many scholars have shown increasing interest in nonnegative irreducible matrices. In 1907, O. Perron [3] discovered some particularly interesting properties of the spectra of positive matrices. G. Frobenius [5], between 1908 and 1912, extended O. Perron's results to the case of nonnegative irreducible matrices and obtained further results. This led to the classical Perron-Frobenius theorem in nonnegative matrix theory, which finds applications in fields such as population dynamics prediction, physics, economics, and others [6]. Subsequent researchers have derived many properties concerning irreducible matrices based on this foundation [5].In 2011, Qi Changjuan [5] delved into the similarity relationship between nonnegative irreducible matrices and positive matrices with a positive trace, particularly when the matrices contained precisely $n+2$ zero elements for n greater than or equal to 8. The study revealed that among this specific class of matrices, the similarity to positive matrices remains undetermined for only six types. However, for all other forms of matrices within this class, it was proven that they can be similar to positive matrices. In 2011, Wang Xinyuan [7] presented a proof that for $n \times n$ nonnegative irreducible matrices containing precisely $n+2$ zero elements and possessing mutually distinct diagonal elements, they can be similar to positive matrices.In 2012, R. Loewy [5] employed a different methodology to establish the validity of the conjecture for four-dimensional matrices. In 2015, Pan Zikang [8] provided proof for specific properties of irreducible matrices and conducted a comprehensive analysis and induction on theorems related to the elements and structures of nonnegative irreducible matrices. Building on the insights of Wang Xinyuan [7], this article provides a proof that when n is greater than or equal to 8, for $n \times n$ nonnegative irreducible matrices containing $n+3$ zero elements and possessing mutually distinct diagonal elements, the matrices can be similar to positive matrices.

2. Basic concept

2.1 Nonnegative (positive) matrix

In the context of matrix theory, a matrix A is classified as a nonnegative (positive) matrix if all its elements are nonnegative (positive) real numbers. Specifically,for any matrix $A = (a_{ij}) \in M_n(R)$,and $i, j = 1,2,\cdots,n. A \geq 0 (A > 0)$ means $a_{ij} \geq 0 (a_{ij} > 0)$ .Therefore, $A \geq 0$ represents A is a nonnegative matrix positive matrix, and $A > 0$ represents a positive matrix.

Nonnegative matrices play a crucial role in various mathematical fields, including linear algebra, graph theory, optimization, and numerical methods. They find applications in modeling processes involving non-negative quantities, such as probabilities, concentrations, and frequencies.

Positive matrices possess unique properties and are essential in numerous fields, including eigenvalue analysis, dynamical systems, and control theory. They are used to model dynamic processes with strictly positive quantities, where the positivity of the elements plays a significant role in the analysis of system stability and convergence.

2.2 Irreducible matrix

For an $n \times n$ square matrix A, if there exists a permutation matrix P such that $P^T AP$ is a block triangular matrix, we refer to the matrix A as reducible; otherwise, we classify the matrix as irreducible. In simpler terms, an irreducible matrix does not possess a block structure or can be reduced to a single connected component through matrix powers. The irreducibility property is a key feature of such matrices, making them essential in the study of dynamical systems, Markov chains, and graph theory, among other areas.

2.3 Symbolic Pattern Matrix

The fundamental idea of Symbolic Pattern in matrix theory is to determine the properties of a matrix solely based on the signs $(+, -, 0)$of its elements. A Symbolic Pattern is a matrix whose elements are drawn from the set $\{+, -, 0\}$, representing positive, negative, and zero values, respectively.

Let A be an n-order nonnegative irreducible sign pattern matrix, where A's elements are either "0" or "+". The n rows and n columns of A are denoted as $A_1, A_2, \cdots, A_n$ and $A^1, A^2, \cdots, A^n$, respectively. If $a_{jk} > 0$, then $a_{ik} > 0$; if $a_{jk} = 0$, then $a_{ik} = 0$ or $a_{ik} > 0$,We say that A_i dominates A_j, denoted as $A_i \geq A_j$. The sum of A_i and A_j, denoted as $A_i + A_j$, is obtained by adding corresponding elements of the i-th and j-th rows of A. The addition satisfies the following rule: if $a_{ik} = a_{jk} = 0$, then $a_{ik} + a_{jk} = 0$; if either a_{ik} or a_{jk} is positive, then $a_{ik} + a_{jk} > 0$. $A_i > 0$ indicates that all elements in the i-th row of A are positive; $A_i \geq 0$ indicates that all elements in the i-th row of Aare nonnegative.$A_i+A_j > 0$ means that all corresponding elements of the i-th and j-th rows of A are positive; $A_i+A_j \geq 0$ means that all corresponding elements of the i-th and j-th rows of A are nonnegative. Similar definitions hold for columns.

3. Theorem

Theorem 1: For any $n \times n$ nonnegative irreducible matrix A, if A contains a row or a column with all positive elements, then A is similar to a positive matrix [9].

Theorem 2: Let P be the zero-nonzero pattern of a nonnegative irreducible matrix. Let $P_i(i = 1,2,\cdots n)$denote the ith row of matrix A, and $P^j(j = 1,\cdots,n)$ denote the jth column of matrix A. If there exist $k,l \in \{1,2,\cdots,n\}$such that $P_k < P_l$ ($P^k \leq P^l$), then any nonnegative irreducible matrix with pattern P is similar to a nonnegative irreducible matrix with pattern Q, where $Q_k = P_l + P_k$, and for $z \neq k$, $Q_z = P_z$ (or $Q^k = P^l + P^k$, and for $z \neq k$, $Q^z = P^z$) [7].

Theorem 3: If A is an $n \times n$ nonnegative irreducible matrix with $n \geq 4$, and A contains exactly n zero elements, then for matrix A, either trA or A is similar to a positive matrix [9].

Theorem 4: Consider a nonnegative irreducible matrix A given by $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} is similar to a positive matrix, and A_{12} contains only positive elements. A is similar to a positive matrix [7].

Theorem 5:For any nonnegative irreducible matrix A,where $a_{ij} = 0$ for $i \neq j$, if $a_{kj} > 0, k \neq i(a_{il} > 0, l \neq j)$,$a_{ii} < a_{jj}\left(a_{jj} < a_{ii}\right)$,, then A is similar to a positive matrix [7].

Theorem 6: For $n \times n$, for any n-order nonnegative irreducible matrix A, if its diagonal elements are distinct and it contains exactly $n + 2$ zero elements, then A is similar to a positive matrix [7].

4. Theorem proving

Theorem: Let A be an $n \times n$ nonnegative irreducible matrix with $n \geq 8$, and it contains n+3 zero elements. If the diagonal elements of A are distinct, then A is similar to a positive matrix.

Proof: By Theorem 1, it is known that for matrix A, each of its rows or columns contains at least one zero element. Therefore, we only need to consider the cases where the nth row of matrix A contains two, three, and four zero elements, respectively.

Case 1: The nth row of A contains four zero elements.

When the diagonal elements of the nth row of A are non-zero, without loss of generality, assume that $a_{n1} = a_{n2} = a_{n3} = a_{n4} = 0$.

First, assume that the zero elements in the first four rows of A do not lie in the same column, which implies that there is at least one column in the first four columns of A containing only one zero element. Without loss of generality, let it be the fourth column. Thus, $A^4 \geq A^1, A^4 \geq A^2$, and $A^4 \geq A^3$. According to Theorem 1, A is similar to a nonnegative irreducible matrix Q, where $Q_1 = A_1 + A_4, Q_2 = A_2 + A_4, Q_3 = A_3 + A_4$,an $Q_k = A_k$for $k \neq 1,2,3$.Consequently, $Q_1 > 0, Q_2 > 0$,and $Q_3 > 0$ by Theorem 1. Applying Theorem 1 again, we conclude that Q is similar to a positive matrix, and therefore, A is similar to a positive matrix.

Next, we consider the case where the zero elements in the first four rows of A lie in the same column.

If the n-th column of A contains only one zero element, we can assume that $a_{15} = a_{25} = a_{35} = a_{45} = 0$. Thus, A is permutation similar to matrix B,

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

Where B_{11} is a k-order matrix, and its pattern is given by:

$$\begin{bmatrix} + & + & + & + & 0 & + & \cdots & + \\ + & + & + & + & 0 & + & \cdots & + \\ + & + & + & + & 0 & + & \cdots & + \\ + & + & + & + & 0 & + & \cdots & + \\ + & + & + & + & + & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & + \\ + & + & + & + & + & \cdots & + & 0 \\ 0 & 0 & 0 & 0 & + & \cdots & + & + \end{bmatrix},$$

Where B_{22} is an $(n-k)$-order nonnegative irreducible matrix containing $n-k$ zero elements, and B_{12} and B_{21} are positive matrices. Since the diagonal elements of B_{22} are distinct, by Theorem Three, it follows that B_{22} is similar to a positive matrix. Furthermore, by Theorem Four, it can be concluded that B is similar to a positive matrix, and therefore A is similar to a positive matrix.

If $a_{1n} = a_{2n} = a_{3n} = a_{4n} = 0$,then A is permutation similar to B

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

Where

$$B_{11} = \begin{bmatrix} + & + & + & + & 0 \\ + & + & + & + & 0 \\ + & + & + & + & 0 \\ + & + & + & + & 0 \\ 0 & 0 & 0 & 0 & + \end{bmatrix},$$

B_{22} is an $(n-4)\times(n-4)$ nonnegative irreducible matrix with $(n-4)$ zero elements. B_{12} and B_{21} are positive matrices. Since A has distinct diagonal elements, B_{22} also has distinct diagonal elements. According to Theorem Three, B_{22} is similar to a positive matrix, and therefore, by Theorem Four, B is similar to a positive matrix.

When the diagonal elements of A in the n-th row are zero, that is, $a_{n1} = a_{n2} = a_{n3} = a_{nn} = 0$.

If there exist $a_{i_1j} = a_{i_2j} = a_{i_3j} = a_{i_4j} = 0$, and $a_{jj} > 0$, then A can be permuted similar to A^T. As shown in the previous proof, A is similar to a positive matrix. If the zeros in A_1, A_2, A_3, A_n appear in the same column, and since $a_{nn} = 0$ and the diagonal elements of A are distinct, we have $a_{n1} = a_{n2} = a_{n3} = a_{nn} = 0$. Since $A^1 \geq A^n$, according to Theorem Two, A is similar to a nonnegative irreducible matrix Q, where $Q_n = A_1 + A_n$, and $Q_k = A_k, k \neq n$. Thus, Q contains n zeros, and by Theorem Three, Ais similar to a positive matrix. If the zeros in A_1, A_2, A_3, A_n do not simultaneously appear in the same column, then there exists $a_{ij} = 0$ for $4 \leq j \leq 7$ and $i \neq j$, where $a_{kj} > 0$ for $k \neq i$ and $a_{il} > 0$ for $l \neq j$. Using Theorem Five, A is similar to a positive matrix.

Case 2: The n-th row of A contains three zero elements.

If there exist elements $a_{i_1j} = a_{i_2j} = a_{i_3j} = a_{i_4j} = 0$, with $a_{jj} > 0$, then A can be permuted to a situation similar to Case 1. If there are no four zero elements simultaneously located in any row or column of the nonnegative irreducible matrix A, we will now discuss this further.

1.When the diagonal elements of the nth row of A are not zero, without loss of generality, assume $a_{n1} = a_{n2} = a_{n3} = 0$. Thus, the matrix pattern of A is as follows:

$$\begin{bmatrix} * & * & * & * & * & * & \cdots & * \\ * & * & * & * & * & * & \cdots & * \\ * & * & * & * & * & * & \cdots & * \\ * & * & * & * & * & * & \cdots & * \\ * & * & * & * & * & * & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & * \\ * & * & * & * & * & \cdots & * & * \\ 0 & 0 & 0 & + & + & \cdots & + & + \end{bmatrix}$$

(1)If the first three columns contain three additional zero elements, then there exist elements $a_{ij} = 0, 4 \leq j \leq n-1, i \neq j$, where $a_{kj} > 0$ for $k \neq i$ and $a_{il} > 0$ for $l \neq j$. By applying Theorem Five, it can be concluded that A is similar to a positive matrix.

(2)If there are two additional zero elements in the first three columns, then at least one column among the first three columns contains one zero element. Without loss of generality, assume that the third column contains one zero element, so we have $A^3 \geq A^2$and $A^3 \geq A^1$. If there exists a zero element in the first two rows that does not coincide with the zero element in the third row, then due to$A^3 \geq A^2$and $A^3 \geq A^1$, according to Theorem One, A is similar to a nonnegative irreducible matrix Q, where $Q_1 = A_1 + A_3, Q_2 = A_2 + A_3$, and $Q_k = A_k$ for $k \neq 1,2$. Consequently, either $Q_1 > 0$ or $Q_2 > 0$, and by applying Theorem One, Q is similar to a positive matrix, which implies that A is similar to a positive matrix.Next, we consider the case where the zero elements in the first two rows coincide with the zero element in the third row in the same column. If $n \geq 8$, then there exist $a_{ij} = 0$ for $4 \leq j \leq n-1$, $i \neq j$, where $a_{kj} > 0$ for $k \neq i$ and $a_{il} > 0$ for $l \neq j$. By applying Theorem Five, it follows that A is similar to a positive matrix.

(3)If the first three columns of A contain an additional zero element, let's assume that A^1 has two zero elements. Therefore, we have $A^2 \geq A^1$ and $A^3 \geq A^1$. If the zero elements in the first row are not in the same columns as the zero elements in the second row (or third row), then due to $A^2 \geq A^1$ ($A^3 \geq A^1$), by applying Theorem One, A is similar to a non-negative irreducible matrix Q, where $Q_1 = A_1 + A_2$ (or $Q_1 = A_1 + A_2$). Consequently, since $Q_1 > 0$, by Theorem One, Q is similar to a positive matrix, and thus A is similar to a positive matrix. If the zero elements in the second row are not in the same column as the zero elements in the third row, and $A^2 \geq A^3$, then A is similar to a non-negative irreducible matrix Q, where $Q_3 = A_2 + A_3$. As $Q_3 > 0$, by applying Theorem One, Q is similar to a positive matrix, and therefore A is similar to a positive matrix. In conclusion, if the zero elements in the first three rows of A are in the same column, then A is similar to a positive matrix; otherwise, it is similar to a positive matrix.

Finally, we consider the case where the zero elements in the first three rows of A are in the same column. If $n \geq 8$, then there exist elements $a_{ij} = 0, 4 \leq j \leq n-1, i \neq j$,such that $a_{kj} > 0$ for $k \neq i$ and $a_{il} > 0$ for $l \neq j$. By applying Theorem Five, it follows that A is similar to a positive matrix.

(4)If the first three columns do not contain any other zero elements, then the matrix pattern of A is as follows:

$$\begin{bmatrix} + & + & + & * & * & * & \cdots & * \\ + & + & + & * & * & * & \cdots & * \\ + & + & + & * & * & * & \cdots & * \\ + & + & + & * & * & * & \cdots & * \\ + & + & + & * & * & * & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & * \\ + & + & + & * & * & \cdots & * & * \\ 0 & 0 & 0 & + & + & \cdots & + & + \end{bmatrix}$$

First, let's consider the case where the zero elements in the first three rows are not in the same column. Since $A^3 \geq A^2$ and $A^3 \geq A^1$, by Theorem Two, A is similar to a nonnegative irreducible matrix Q, where Q contains k zero elements. Since $k \leq n+2$, by Theorem Six, Q is similar to a positive matrix, and thus A is similar to a positive matrix.

Next, let's consider the case where the zero elements in the first three rows are not in the same column, and let's assume $a_{1r} = a_{2r} = a_{3r} = 0$. If $r = n$, then A is permutation-similar to matrix B,

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

Where

$$B_{11} = \begin{bmatrix} + & + & + & 0 \\ + & + & + & 0 \\ + & + & + & 0 \\ 0 & 0 & 0 & + \end{bmatrix}$$

In the first case, where B_{22} is an $(n-4)\times(n-4)$ nonnegative irreducible matrix containing $(n-4)$ zero elements, and B_{12} and B_{21} are positive matrices, it follows from Theorem Three that B_{22} is similar to a positive matrix. Then, using Theorem Four, we conclude that B is similar to a positive matrix, and consequently, A is also similar to a positive matrix.

For the second case, if $4 \leq r \leq n-1$, there exists $a_{ij} = 0, 4 \leq j \leq n-1, i \neq j$,such that $a_{kj} > 0$ for $k \neq i$ and $a_{il} > 0$ for $l \neq j$.By applying Theorem Five, we can deduce that A is similar to a positive matrix.

2.Similarly ,we can prove the case When the diagonal elements of A in the nth row are zero.

Similarly, we can prove the case where A's nth row contains two zero elements.

5. Future Outlook

Matrix theory finds extensive applications not only within various mathematical disciplines, such as combinatorics, graph theory, number theory, algebra, polynomials, finite geometry, operations research, and statistics, but also beyond mathematics in fields like control theory, systems theory, information theory, signal processing, and economics [10,11]. With the continuous advancement of computer and communication technologies, the practical use of matrix theory in modern engineering has become increasingly widespread [12]. Matrix theory serves as both a useful tool and a vibrant, vast research area. Among the active research domains in linear algebra, nonnegative irreducible matrix theory stands out, finding broad applications in many branches of mathematics, natural sciences, social sciences, and beyond [1].

Nonnegative irreducible matrices that are similar to positive matrices form a significant class of matrices, and their study holds both theoretical and practical value. From a theoretical perspective, investigating the properties of nonnegative irreducible matrices similar to positive matrices deepens our understanding of matrix structures and characteristics, providing novel insights and methods for the advancement of matrix theory.

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