Effective Quaternion Rotation in An Ellipsoid through Sphere Transformation: A Linear Approach

Rundong Li [1,a], Muwen Chen [1,b]

[1] Wenzhou-Kean University, Wenzhou 325000, Zhejiang Province, China.

[a] lirund@kean.edu, [b] chenmu@kean.edu

Abstract. The application of quaternion rotation in ellipsoids and spheres is an intriguing field with significant implications in computer graphics, robotics, and physics simulations. The necessity to decipher quaternion rotation in ellipsoid is crucial. This research aims to utilize the principles of quaternion rotation in spheres to compute the equivalent in the ellipsoid. This involves the application of a linear transformation to convert quaternion rotation in ellipsoids into spheres. By controlling the full rotation of quaternions, the spheres can be transformed back into ellipsoids, achieving the ultimate goal of controlling quaternion rotation within an ellipsoid. The results demonstrate that this approach effectively addresses the quaternion rotation on the ellipsoid and aligns with the fundamental properties of quaternions. Furthermore, it serves as a significant aid in implementing quaternion rotation on the ellipse.

Keywords: Quaternion rotation; Ellipsoid; Computer graphics.

1. Introduction

Quaternion rotation in ellipsoids is a topic of great interest in computer graphics, robotics, and physics simulations. As three-dimensional geometric objects, Ellipsoids play a crucial role in various applications such as shape modeling, animation, and collision detection. The ability to accurately and efficiently rotate ellipsoids is essential for realistic and interactive simulations.

Traditional rotation representations, such as Euler angles and rotation matrices, have limitations when describing rotations intuitively and efficiently. Euler angles suffer from gimbal lock, a phenomenon where specific orientations lead to a loss of degrees of freedom, resulting in ambiguous rotations. On the other hand, rotation matrices are prone to computational complexities and are less compact in terms of storage.

In recent years, quaternion-based rotation has emerged as a powerful alternative for representing and manipulating rotations. Using unit quaternions is a more convenient and concise way to represent rotations than rotation matrices because the inverses of unit quaternions can be easily computed and have good mathematical properties, providing a more compact representation than rotation matrices. They avoid the issues associated with gimbal locks and offer compact storage, making them suitable for real-time applications.

This research paper aims to investigate the use of quaternions for rotating ellipsoids. The primary objective is to explore the benefits and advantages of employing quaternion-based rotations in the context of ellipsoids, focusing on their efficiency, accuracy, and versatility. By leveraging the mathematical properties of quaternions, we can achieve robust and intuitive ellipsoid rotations.

The paper will discuss the fundamental principles of quaternions, including their mathematical definition, algebraic operations, and conversion to rotation matrices. Additionally, the specific formulation for rotating ellipsoids using quaternions will be presented, elucidating the transformation process and the effects on the ellipsoids' shape and orientation. We will also discuss ellipses' rotation matrix and quaternions' rotation and behavior on ellipsoids.

2. Methodology

2.1 Quaternion rotation on ellipsoid

This paper aims to investigate the application of quaternions to the rotation of ellipsoids. Initially, the rotation of geometric figures using quaternions was only applied to unit spheres. Thus, the fundamental concept of quaternion rotation on an ellipsoid involves linearly transforming the ellipsoid into a sphere. The standard equation for an ellipsoid centered at the origin is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{1}$$

The quaternion on the ellipsoid (satisfying the standard equation of the ellipsoid) is:

$$v=0+\alpha i+\beta j+\gamma k \tag{2}$$

Its coordinates are:

$$v=\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} \tag{3}$$

A linear transformation is performed on each point on the ellipsoid, including the quaternion, as follows:

$$x^{'}=\frac{x}{a}, y^{'}=\frac{y}{b}, z^{'}=\frac{z}{c} \tag{4}$$

$$v'=\begin{bmatrix}\frac{\alpha}{a}\\ \frac{\beta}{b}\\ \frac{\gamma}{c}\end{bmatrix} \tag{5}$$

This results in the new spherical equation:

$$x^2+y^2+z^2=1 \tag{6}$$

The quaternion controlling the rotation is:

$$q=q_0+q_1 i+q_2 j+q_3 k \tag{7}$$

Here, 'q' adheres to the basic rules of quaternion rotation control on a sphere, i.e., the real part controls the rotation angle, and the imaginary part controls the rotation axis. Moreover, before the rotation, it is necessary to normalize the axis vector, making it a unit quaternion, to comply with the basic elements of sphere rotation:

$$q=q_0+\frac{q_1}{\sqrt{{q_1}^2+{q_2}^2+{q_3}^2}}i+\frac{q_2}{\sqrt{{q_1}^2+{q_2}^2+{q_3}^2}}j+\frac{q_3}{\sqrt{{q_1}^2+{q_2}^2+{q_3}^2}}k \tag{8}$$

Here we employ the quaternion JPL rotation matrix:

$$R=\begin{bmatrix}1-2q_2^2-2q_3^2 & 2q_1q_2+2q_0q_3 & 2q_1q_3-2q_0q_2\\ 2q_1q_2+2q_0q_3 & 1-2q_1^2-2q_3^2 & 2q_2q_3+2q_0q_1\\ 2q_1q_3+2q_0q_2 & 2q_2q_3-2q_0q_1 & 1-2q_1^2-2q_2^2\end{bmatrix} \tag{9}$$

The rotation is carried out through the quaternion rotation matrix:

$$v'_{rotated} = Rv' \tag{10}$$

This yields the vector in the sphere after the ellipsoid's linear transformation. Next, it is transformed back into the ellipsoid:

$$v_{rotated} = \begin{bmatrix} \alpha'_{rotated} * a \\ \beta'_{rotated} * b \\ \gamma'_{rotated} * c \end{bmatrix} \tag{11}$$

Upon reverse deduction, the rotation matrix for the quaternion on the ellipsoid can be obtained:

$$R_{ellipsoid} = \begin{bmatrix} 1 - 2q_2^2 - 2q_3^2 & \frac{2a}{b}(q_1q_2 + q_0q_3) & \frac{2a}{c}(q_1q_3 - q_0q_2) \\ \frac{2b}{a}(q_1q_2 + q_0q_3) & 1 - 2q_1^2 - 2q_3^2 & \frac{2b}{c}(q_2q_3 + q_0q_1) \\ \frac{2c}{a}(q_1q_3 + q_0q_2) & \frac{2c}{b}(q_2q_3 - q_0q_1) & 1 - 2q_1^2 - 2q_2^2 \end{bmatrix} \tag{12}$$

The properties of the derived rotation matrix for the quaternion on the ellipsoid are the same as those of the basic quaternion rotation matrix, and they can be used to rotate the quaternion on the ellipsoid.

2.2 Quaternion rotation on ellipse

The basic equation of the ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{13}$$

The quaternion controlling the rotation on the ellipse is:

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} * v \tag{14}$$

The rotation axis vector is fixed as follows:

$$v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{15}$$

This ensures that the rotation takes place on the equatorial plane of the ellipsoid, i.e., the ellipse. Thus, it is possible to control the rotation of the quaternion on the ellipse by controlling the size of θ, and the derived rotation matrix for the quaternion on the ellipse is:

$$R_{ellipse} = \begin{bmatrix} 1 - 2sin\theta^2 & \frac{2a}{b}cos\frac{\theta}{2}sin\frac{\theta}{2} & 0 \\ \frac{2b}{a}cos\frac{\theta}{2}sin\frac{\theta}{2} & 1 - 2sin\frac{\theta}{2}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

3. Literature Review

Quaternion is a mathematical structure that extends a complex number consisting of one real part and three imaginary parts. In mathematics, a quaternion is a vector in a four-dimensional space over the field of real numbers (denoted as "w"). The quaternion space, represented by Q, has an ordered basis consisting of R, i, j, and k [1]. A quaternion can be expressed as $\mathrm{Q} = \mathrm{q}_0 + \mathrm{q}_1\mathrm{i} + \mathrm{q}_2\mathrm{j} + \mathrm{q}_3\mathrm{k}$, and $\mathrm{q}_0, \mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3$ are the real part, $\mathrm{i}, \mathrm{j}, \mathrm{k}$ are the imaginary part of the quaternion unit. The imaginary part of the quaternion unit has the following properties: $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$, which shows that the imaginary part of a quaternion satisfies a non-commutative and non-associative multiplication rule. For the multiplication of quaternions, it can be defined as a pair $(\mathrm{Sq}, \mathrm{Vq})$, where $\mathrm{Sq} = \mathrm{a0} \in \mathrm{R}$ represents the scalar part of the quaternion, and $\mathrm{Vq} = \mathrm{a1i} + \mathrm{a2j} + \mathrm{a3k} \in \mathrm{R}^3$ represents the vector part of q. The quaternion product of two quaternions, p, and q, is defined as $pq = Sp * Sq - < Vp, Vq > + SpVq + SqVp + Vp \wedge Vq$ [2].

Quaternions have many applications in 3D space, especially in robotics and aerospace engineering. Caccavale and Siciliano employed quaternions to prevent representational singularities and devised kinematic control strategies for a redundant manipulator installed on a freely-floating spacecraft [3]. Islam, Okasha, and Sulaeman find that employing a unit quaternion-based quadcopter with MPC enables smooth and singularity-free flight [4]. Yang [5] discusses a quaternion-based approach for spacecraft modeling, attitude determination, and control. It has significant advantages in spacecraft attitude measurement, actuator and disturbance torque, attitude modeling, determination, estimation, and attitude control techniques [5]. Quaternions accelerate rotation-related computations, reduce storage space, prevent numerical inaccuracies caused by floating-point calculations, and facilitate interpolation in keyframe animations in computer graphics [6]. Goldman [6] rethinks quaternions and highlights their practicality in robot kinematics for deriving forward and inverse kinematics of robotic arms. Chen and Hung [7] present the application of quaternions in robot control, offering improved computational efficiency, reduced overhead, and coordinate invariance. Quaternions provide an efficient way to describe and process complex motions and transformations in 3D space, enabling many 3D-related technologies and applications to be realized.

Compared with other mathematical tool, such as the Euler angle and transformation matrix, quaternion also have many advantages. Compared with the Euler angle, quaternions can be easily interpolated linearly to achieve smooth rotation transitions. The research by Challis shows that the clear definition of rigid body orientation in three dimensions using quaternions, along with its straightforward averaging and interpolation techniques, makes it highly practical for the kinematic analysis of human motion [8]. Compared with the rotation matrix, quaternion multiplication is faster, making it better for real-time computer graphics and animation applications. Evans and Murad represented direction by using quaternions and use the method of fifth-order predictive corrector to solve the Euler rigid body equation. Compared to the previous approach, this algorithm enhances computational speed tenfold [7]. Combined with the above advantages, quaternions are widely used in 3D rotation representation and attitude transformation and have more advantages than other mathematical tools. Quaternions show their unique value in computer graphics, robotics, and aerospace engineering.

Additionally, dual quaternions, extending traditional quaternions with extra dual imaginary components, show great potential in robotics, computer graphics, and animation, offering multiple advantages, including enhanced computational efficiency, reduced overhead, and coordinate invariance [10]. Kenwright [10] extensively explores dual quaternions, their applications, and their significance from classical mechanics to computer graphics and beyond.

4. Result and Discussion

Through the linear transformation of fundamental quaternions, this paper accomplishes the control of quaternion rotation on an ellipsoid, filling the gap where quaternion rotation could not be implemented on other surfaces. Based on a sphere, this ellipsoid quaternion shares similar properties with basic quaternions, including its rotation method and path. Furthermore, the article also discusses how to perform quaternion rotation on an ellipse, indicating that in the future, quaternions can better control effects in 3D graphics.

In achieving quaternion rotation control on an ellipsoid, this paper explores the broader implications of quaternion transformations and their applicability in computer graphics, robotics, and physics simulations. The findings indicate the possibility of overcoming traditional hindrances in quaternion rotation and extending their capabilities to more diverse surfaces. The discussion on performing quaternion rotation on an ellipse opens the door to future improvements in the precision and quality of 3D imaging, broadening the range of quaternion's potential applications.

However, there are still some limitations, such as irregular curves. It should be noted that while promising, the methodology still finds challenges in irregular curves, demonstrating a need for further research. Hence, future studies could build on this work by devising more sophisticated linear transformation techniques to address more complex surface rotations, thereby broadening the scope and capabilities of quaternion rotation applications. Applying linear transformations to address more complex surface rotation issues may be an important research direction for the future.

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