Testing fuzzy hypotheses with crisp data based on p-value

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Abstract. In this paper, we propose the p-value of fuzzy hypothesis by the concept of projection of the fuzzy relation, the small p-value gives evidence that research hypothesis is true. For the crisp hypothesis, we give the fuzzy set description of the test and prove that the rejection region of the test is the cut set of the fuzzy set. Finally, the p-values of fuzzy hypothesis are given by specific examples.

Keywords: fuzzy hypothesis; p-value; significance testing.

1. Introduction

Statistical inference is the procedure by which researchers and practitioners draw conclusions about parameters of interest using data from a representative sample. Hypothesis testing is kown to one of the crucial forms of statistical inference and the most frequent statistical method widely used by scientists to guard against making claims unjustified by data. The null hypothesis and the research hypothesis are usually described as two crisp subsets of the parameter space in classical statistical hypothesis testing. But the crisp hypotheses may not fit well with the conclusions that people are actually trying to verify. For example, it maybe not exactly to convey appropriate semantic information such as "about", "around", "close", "short", among others.

However, fuzzy hypotheses provide suitable ways to address such imprecisions and uncertainties [1, 2]. For instance, suppose θ is the proportion of defective parts produced by a factory. We're going to take a random sample and make some inferences about θ from the sample. In crisp hypothesis testing, the form of hypothesis " $H_0 : \theta = 0.03 \leftrightarrow H_0 : \theta \neq 0.03$ ". or " $H_0 : \theta = 0.03 \leftrightarrow H_1 : \theta > 0.03$ " are considered; and so on. In practice, more realistic expressions about θ would be considered as: small, approximately, and so on. Therefore, more realistic formulation of hypotheses might be: "$H_0 : \theta$ is near to 0.03", versus "$H_1 : \theta$ is away from 0.03". We call such expressions as fuzzy hypotheses. It is easy to see that crisp hypotheses is a special case of fuzzy hypotheses, simply replace membership function of the fuzzy hypotheses with the indicator function of crisp hypotheses.

In recent decades, testing fuzzy statistical hypotheses have been proposed by many researchers. Arnold [1, 2] worked on testing fuzzy hypotheses with crisp data, gave new definitions for probability of type I and type II errors and presented a best test for the one-parameter exponential family. Tanaka et al. [3] investigated hypotheses testing problem with fuzzy data in the decision problem framework. Taheri and Behboodian [4] formulated the problem of testing fuzzy hypotheses when the hypotheses are fuzzy and the observations are crisp. Haktanır et al. [5] used Z-fuzzy numbers to capture the vagueness in the sample data and develop Z-fuzzy hypothesis testing. Torabi and Behboodian [6] recalled and redefined some concepts about fuzzy hypotheses testing, and then introduced the likelihood ratio test for fuzzy hypotheses testing. Yosefi et al [7] proposed a new approach for testing fuzzy hypotheses based on likelihood ratio statistic. Arefi and Taheri [7, 9] proposed a new approach for testing fuzzy parametric hypotheses based on fuzzy test statistic. As an important form of hypothesis testing results, testig fuzzy hypotheses based on p-value is not uncommon. Parchami et al. [10, 11] dealed with the problem of testing statistical hypotheses when both the hypotheses and data are fuzzy and introduced the concept of fuzzy p-value, developed an approach for testing fuzzy hypotheses by comparing a fuzzy p-value and a fuzzy significance level.

Based on p-value, a procedure is illustrated to test various types of fuzzy hypotheses with crisp data by Parchami [12], in which, the p-value of the fuzzy hypothesis is obtained from the weighted average of the classical p-value using membership function corresponding to the fuzzy hypothesis. In this paper we are going to present a new p-value-based approach for testing fuzzy hypotheses. The proposed method is on the basis of the concept of the projection of the fuzzy relation which determined by p-value function. Also, it must be mentioned that all results of this study coincide to the results of testing classical hypotheses, when the fuzzy hypotheses reduce to crisp hypotheses.

This paper is organized as follows. Section 2 recalls the classical p-value function and hypotheses test. In Section 3, starting from fuzzy sets and fuzzy relations, we construct and explain the p-value of fuzzy hypotheses from the projection of the fuzzy relation. In Section 4, we calculate the p-values of fuzzy hypothesis tests for several different distributions. A conclusion is given in the final section.

2. The p-value of the hypothesis test

A statistical hypothesis is a statement about one or more population distributions or their parameters, which are simply called hypothesis. Hypothesis testing problem usually raises two complementary hypotheses, null hypothesis and research hypothesis or alternative hypothesis, which are denoted H_0 and H_1 respectively. For parametric hypothesis testing problems, the range Θ , of the unknown parameter θ is known, is the parameter space, and it is assumed that Θ can be divided into two disjoint parts Θ_0 and Θ_1 .We denote H_0 for the hypothesis $\theta \in \Theta_0$ and H_1 for the hypothesis $\theta \in \Theta_1$. Obviously, there is one and only one correct H_0 and H_1 .

For a hypothesis testing problem $H_0 : \theta \in \Theta_0 \leftrightarrow H_1 : \theta \in \Theta_1$, a testing rule (referred to as the test) is to divide the sample space χ of sample $(x_1, x_2, \cdots, x_n)$ into two disjoint subsets W and W^c . When sample $(x_1, x_2, \cdots, x_n) \in W$, reject null hypothesis H_0 (i.e., accept H_1). Such a partition of the sample space constitutes a testing rule, and the subset W is called the rejection region or critical region of the test.

The vital reported result of hypothesis testing is p-value and its informal definition is given in the ASA statement[13]: a *p*-value is the probability under a specified statistical model that a statistical summary of the data (e.g., the sample mean difference between two compared groups) would be equal to or more extreme than its observed value. This tells us that the p-value has two elements: the sample observations and the hypothesis distribution. The value of the test statistic is calculated from the sample observations, and then the corresponding p-value is calculated and determined from the hypothesis distribution.

Definition 1 Let sample $\mathbf{X}$ come from the population with the parameter θ , and $T(\mathbf{X})$ is a test statistic. Assume that the larger the value of $T(\mathbf{X})$ is, the more detrimental it is to H_0 . Then for a sample observation $\mathbf{x}$, the p-value function of the test is

$$p(\mathbf{x}, \theta) = P_\theta \left\{ T(\mathbf{X}) \geq T(\mathbf{x}) \right\}. \tag{1}$$

Obviously, the p-value function is a binary function of observations and parameters. The p-value function here neither need to be associated with a particular null hypothesis, nor the probability that the null hypothesis is true.

3. Testing fuzzy hypotheses with crisp data based on p-value function

3.1 Fuzzy set and fuzzy relation

Fuzzy set, a generalization of classical crisp set, were created by Zadeh [15]. Over the past 50 years, great achievements have been made in fuzzy set methods and extensions, both in theory and in application fields. Fuzzy set uses membership function to reflect the degree to which an element belongs to the set. Through reasonable construction, membership function can be used to describe the degree of agreement between the sample value and the null hypothesis.

Definition 2 [15] Let U is a universe set, a map A from U to interval $[0,1]$ is a fuzzy set on U , i.e.

$$A: U \to [0,1],\ u \mapsto A(u).$$

$A(u)$ is called membership degree of element u to the fuzzy set A . For $0<\alpha<1$, the classical set $A_\alpha = \{u \in U \mid A(u) \geq \alpha\}$ is called the α -cut set of A , $A_{\overline{\alpha}} = \{u \in U \mid A(u) > \alpha\}$ is called the α -strong cut set of A . The entire fuzzy set on U is denoted as $\mathcal{F}(U)$.

Definition 3 Let U , V are two universe sets, fuzzy set R on $U \times V$ is called the fuzzy relation from U to V , i.e.

$$R: U \times V \to [0,1],\ (u,v) \mapsto R(u,v).$$

$R(u,v)$ denotes the degree to which u and v are related R . Also called mapping

$$f: U \to \mathcal{F}(V),\ u \mapsto f(u) \in \mathcal{F}(V)$$

is a fuzzy map from U to V .

Clearly, such $R \in \mathcal{F}(U \times V)$, a fuzzy mapping from U to V can be uniquely determined: $f_R(u)(v) = R(u,v)$. For a given element $u_0 \in U$, fuzzy set $f_R(u_0)$ is called the projection of R at u_0 , denoted R_{u_0} . Conversely, any given map $f: U \to \mathcal{F}(V)$, the fuzzy relation $R_f(u,v) = f(u)(v)$ can be uniquely determined from U to V .

3.2 Testing fuzzy hypotheses based on p-value function

As mentioned in introduction, the fuzzy hypothesis may be a more fitting representation of the conclusion one really wants to test. This can be achieved by replacing the crisp subset with a fuzzy subset.

Definition 4 [12] Any hypothesis of the form " $H: \theta$ is $\tilde{\Theta}$ " is called a fuzzy hypothesis, where implies that θ is in a fuzzy subset $\tilde{\Theta}$ of Θ (the parameter space) with membership function $H(\theta)$ i.e. a function from Θ to $[0,1]$.

Note that the crisp hypothesis $H_i: \theta \in \Theta_i$ is a fuzzy hypothesis with membership function $H(\theta)=1$ at $\theta \in \Theta_i$, and zero otherwise, i.e., the indicator function of the crisp set Θ_i , $i = 0,1$.

Example 1 Let θ be the parameter of a binomial distribution. Consider the following function

$$H_0(\theta) = \begin{cases} 2\theta, & 0<\theta<0.5, \\ 2-2\theta, & 0.5<\theta<1. \end{cases}$$

The hypothesis " $H_0: \theta$ is $\tilde{\Theta}_0$ " is a fuzzy hypothesis with membership function $H_0(\theta)$ and it means that $\theta \simeq 0.5$, i.e. " θ is approximately 0.5 " (see Fig. 1).

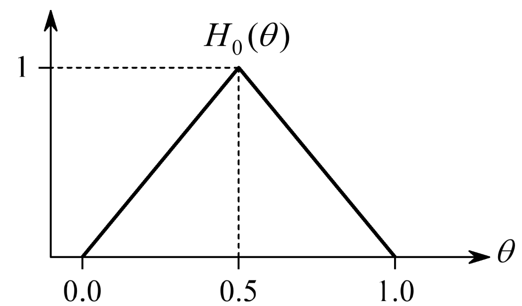

Fig. 1 The membership functions of the fuzzy hypotheses in Example 1

In the following, based on the p-value function $p(\mathbf{x},\theta)$ (Definition 1) of hypothesis testing, we construct a p-value of fuzzy hypotheses. $p(\mathbf{x},\theta)$ is a binary function of the observation **x** and the parameter θ, with the value between 0 and 1. It can be regarded as a fuzzy relation on the domain $\mathbb{X} \times \Theta$.

Definition 5 Let $p(\mathbf{x},\theta)$ is p-value function and **x** is observed data, with a slight abuse of notation, a fuzzy subset on universe $\mathbb{X}$

$$p_{\tilde{\Theta}_0} \in \mathcal{F}(\mathbb{X}) : \mathbf{x} \mapsto p_{\tilde{\Theta}_0}(\mathbf{x}) = \bigvee_{\theta\in\Theta} p(\mathbf{x},\theta)\cdot H_0(\theta), \tag{2}$$

is called p-value of fuzzy hypothesis H_0.

Remark When the fuzzy hypotheses degenerate into the classical crisp hypotheses, we know the membership function of Θ_0 is an indicator function, so $p_{\Theta_0}(\mathbf{x}) = \bigvee_{\theta\in\Theta_0} p(\mathbf{x},\theta) = \sup_{\theta\in\Theta_0} p(\mathbf{x},\theta)$ is classical p-value, we can call it the fuzzy set description of the test.

From the Definition 5, $p_{\tilde{\Theta}_0}$ is the projection of fuzzy relation $p(\mathbf{x},\theta)$ onto fuzzy set $\tilde{\Theta}_0$, represents the degree of confidence in the null hypothesis for different sample values, is the membership function of "sample value and the null hypothesis degree of agreement". Similar to classical hypothesis testing, $p_{\tilde{\Theta}_0}$ is the observed significance level of fuzzy hypothesis H_0, the small values of $p_{\tilde{\Theta}_0}$ give evidence that H_1 is true.

Theorem 1 For the crisp hypothesis $H_0 : \theta \in \Theta_0$, p_{Θ_0} is a valid p-value, i.e.

$$P_\theta\left\{p_{\Theta_0}(\mathbf{X}) \le \alpha\right\} \le \alpha,\ \forall\theta\in\Theta.$$

To prove theorem 1, we first introduce the following conclusion.

Lemma 1 Let $F(x)$ be the cumulative distribution function of a random variable X. Let $Y = F(X)$, for all $0 < y < 1$, we have $P(Y \le y) \le y$.

Proof: For $0 < y < 1$, let $A_y = \{x : F(x) \le y\}$, considering that $F(x)$ is monotonically nondecreasing, A_y is a semi-infinite region between $(-\infty, x_y)$ or $(-\infty, x_y]$. When $A_y = (-\infty, x_y]$, there is

$$P\{Y \le y\} = P\{F(X) \le y\} = P\{X \in A_y\} = F(x_y) \le y.$$

When $A_y = (-\infty, x_y)$,

$$P\{Y \le y\} = P\{F(X) \le y\} = P\{X \in A_y\} = P\{X < x_y\} = \lim_{\delta\downarrow 0} P\{X \le x_y - \delta\},$$

For any $\delta > 0$, $x_y - \delta \in A_y$, thus $P\{X \le x_y - \delta\} = F(x_y - \delta) \le y$, so

$$P\{Y \le y\} = \lim_{\delta\downarrow 0} P\{X \le x_y - \delta\} \le y.$$

In summary, the reasoning is proved.

The proof of Theorem 1: For a fixed $\theta \in \Theta$, let $F_\theta(t)$ denote the cumulative distribution function of $-T(\mathbf{X})$, then

$$p(\mathbf{x},\theta) = P_\theta\left\{T(\mathbf{X}) \ge T(\mathbf{x})\right\} = P_\theta\left\{-T(\mathbf{X}) \le -T(\mathbf{x})\right\} = F_\theta(-T(\mathbf{x})).$$

Thus $p(\mathbf{X},\theta) = F_\theta(-T(\mathbf{X}))$, and for $\forall\ 0 < \alpha < 1$, Lemma 1 shows that

$$P_\theta\{p(\mathbf{X},\theta) \le \alpha\} = P_\theta\{F_\theta(-T(\mathbf{X})) \le \alpha\} \le \alpha.$$

And $\left\{p_{\Theta_0}(\mathbf{X}) \le \alpha\right\} \subseteq \left\{p(\mathbf{X},\theta) \le \alpha\right\}$, $\forall\theta\in\Theta$, so $P_\theta\left\{p_{\Theta_0}(\mathbf{X}) \le \alpha\right\} \le \alpha$. Theorem 1 is proved.

Theorem 2 For the fuzzy hypothesis $H_0 : \theta \in \tilde{\Theta}_0$, if the membership function $H_0(\theta)$ satisfies compatibility condition: $\sup_{\theta\in\Theta} p(\mathbf{x},\theta)\cdot H_0(\theta) \geq p(\mathbf{x},\theta)$, then $p_{\tilde{\Theta}_0}$ is a valid p-value.

The proof is similar to Theorem 1.

Analogy to the classical hypothesis testing, the rejection region of the fuzzy hypothesis can be defined as follows.

Definition 6 For a fuzzy hypothesis testing $H_0 : \theta \in \tilde{\Theta}_0 \leftrightarrow H_1 : \theta \in \tilde{\Theta}_1$, each $0 < \alpha < 1$,

$$(p_{\tilde{\Theta}_0})^c_{\bar{\alpha}} = \{\mathbf{x} \in \mathrm{X} \mid p_{\tilde{\Theta}_0}(\mathbf{x}) \leq \alpha\} \tag{3}$$

is called rejection region of α -level testing.

Theorem 3 $p_{\tilde{\Theta}_0}(\mathbf{x}) = \inf\left\{\alpha \in [0,1] \middle| \mathbf{x} \in (p_{\tilde{\Theta}_0})^c_{\bar{\alpha}}\right\}$.

Proof: $\inf\left\{\alpha \in [0,1] \middle| \mathbf{x} \in (p_{\tilde{\Theta}_0})^c_{\bar{\alpha}}\right\} = \inf\left\{\alpha \middle| p_{\tilde{\Theta}_0}(\mathbf{x}) \leq \alpha\right\} = p_{\tilde{\Theta}_0}(\mathbf{x})$.

Theorem 3 shows that $p_{\tilde{\Theta}_0}(\mathbf{x})$ is the minimum significance level at which the test makes a reject decision of the null hypothesis given the sample $\mathbf{x}$.

3.3 Numerical examples

In this section, we illustrate different situations of testing fuzzy hypotheses by three examples.

Example 2 The manager of a factory thinks that the average current consumption of the small motors produced by the factory will not exceed 0.8A under normal load conditions. It is found that the average current consumed by 16 motors is 0.92A, and the standard deviation of these 16 samples is 0.32A. Assuming that the current consumption X of the motor follows a normal distribution $N(\theta,\sigma^2)$, we consider the fuzzy hypotheses“ H_0 : θ is approximately smaller than or equall to 0.8”, versus “$H_1 : \theta$ is approximately bigger than 0.8”. Where H_0 and H_1 have the following membership function (see Fig. 2)

$$H_0(\theta) = \begin{cases} 1, & 0 \leq \theta \leq 0.7, \\ -5\theta + 4.5, & 0.7 < \theta < 0.9, \\ 0, & \theta \geq 0.9, \end{cases} \qquad H_1(\theta) = \begin{cases} 0, & 0 \leq \theta \leq 0.6, \\ 5\theta - 3, & 0.6 < \theta < 0.8, \\ 1, & \theta \geq 0.8. \end{cases}$$

Considering a sufficient statistic, $T = \frac{\bar{X} - \theta}{S}\sqrt{n} \sim t(n-1)$, according to Definition 5, we can get the following p-value of fuzzy hypotheses H_0,

$$\begin{aligned} p_{\tilde{\Theta}_0}(\mathbf{x}) &= \vee_{\theta\in\Theta} \{P(T > t)\cdot H_0(\theta)\} \\ &= \vee_{\theta\in\Theta} \left\{P\left(T > \frac{0.92-\theta}{0.32}\sqrt{16}\right)\cdot H_0(\theta)\right\} \\ &= \vee_{\theta\in\Theta} \left\{\left(1 - \mathrm{pt}\left(\frac{0.92-\theta}{0.08}, 15\right)\right)\cdot H_0(\theta)\right\} \\ &= 0.0500. \end{aligned}$$

where pt() is the distribution function of t distribution.

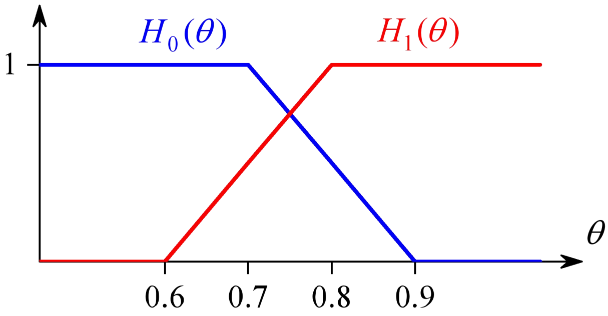

Fig. 2 The membership functions of the fuzzy hypotheses in Example 2

Example 3 The manager of a factory reinstalled a new system to improve the safety of his employees. We can assume that the average of the number of accidents per month obeys a Poisson distribution with parameter θ . A study showed that 27 accidents occurred in the plant during the past year. After installing the new system, the manager wants to test whether the monthly average of accidents is approximately bigger 3. Then we should test hypotheses " $H_0 : \theta$ is approximately bigger than 3", versus " $H_1 : \theta$ is approximately smaller than 3". Where H_0 and H_1 have the following membership function (see Fig. 3)

$$H_0(\theta)=\begin{cases}0, & \theta<2.75,\\ 2(\theta-2.75), & 2.75<\theta\le 3.25,\\ 1, & \theta\ge 3.25,\end{cases}\quad H_1(\theta)=\begin{cases}1, & \theta<2.75,\\ 2(3.25-\theta), & 2.75\le\theta<3.25,\\ 0, & \theta\ge 3.25.\end{cases}$$

Therefore, considering a sufficient statistic, $T=\sum_{i=1}^{12} X_i \sim P(12\theta)$, we can get the following p-value,

$$\begin{aligned} p_{\tilde{\Theta}_0} &= \vee_{\theta\in\Theta}\{P(T\le t)\cdot H_0(\theta)\} \\ &= \vee_{\theta\in\Theta}\{P(T\le 27)\cdot H_0(\theta)\} \\ &= \vee_{\theta\in\Theta}\left\{\sum_{t=0}^{27}\frac{e^{-12\theta}(12\theta)^t}{t!}\cdot H_0(\theta)\right\} \\ &= 0.0369. \end{aligned}$$

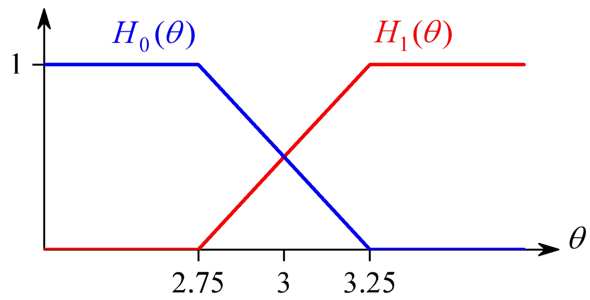

Fig. 3 The membership functions of the fuzzy hypotheses in Example 3

Example 4 One kind of screw is produced in a factory, the standard length is 68mm, its length X obeys normal distribution $N(\theta, 3.6^2)$, 36 samples are randomly selected and the average is 68.5 mm. considering testing fuzzy hypothesis " $H_0 : \theta$ is near to 68", versus " $H_1 : \theta$ is away from 68". Where H_0 and H_1 have the following membership function (see Fig. 4)

$$H_0(\theta)=\begin{cases}\theta-67, & 67\le\theta\le 68,\\ -\theta+69, & 68<\theta\le 69,\\ 0, & \text{otherwise},\end{cases}\quad H_1(\theta)=\begin{cases}-\theta+68, & 67\le\theta\le 68,\\ \theta-68, & 68<\theta\le 69,\\ 1, & \text{otherwise}.\end{cases}$$

As we all known, $T=\dfrac{\bar{X}-\theta}{0.6}\sim N(0,1)$, T is a sufficient statistic, according to the principle of sufficiency, and the larger $|T|$ is, the less favorable to the null hypothesis. We can obtain the following p-value function of fuzzy hypothesis " $H_0:\theta$ is near to 68", versus " $H_1:\theta$ is away from 68",

$$\begin{aligned}p_{\tilde{\Theta}_0}&=\underset{\theta\in\Theta}{\vee}\left\{P(|T|\ge|t|)\cdot H_0(\theta)\right\}\\&=\underset{\theta\in\Theta}{\vee}\left\{P\left(|T|\ge\left|\frac{68.5-\theta}{0.6}\right|\right)\cdot H_0(\theta)\right\}\\&=\underset{\theta\in\Theta}{\vee}\left\{2\left(1-\Phi\left(\left|\frac{68.5-\theta}{0.6}\right|\right)\right)\cdot H_0(\theta)\right\}\\&=0.5220.\end{aligned}$$

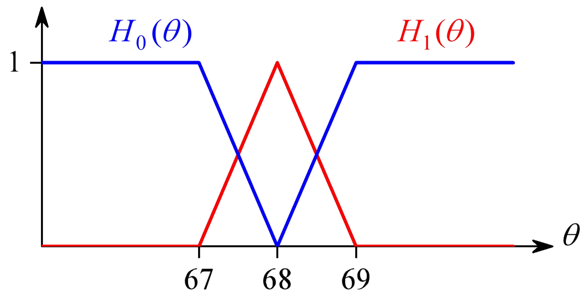

Fig. 4 The membership functions of the fuzzy hypotheses in Example 4

4. Summary

In this paper, the p-value function is interpreted as the fuzzy relation between the sample observations and the parameter space, and the p-value of the fuzzy hypothesis is constructed by the projection of the fuzzy relation onto fuzzy hypothesis, which describes evidence of the sample values to the null hypothesis, and quantitatively describes the degree of support for the null hypothesis of the sample observations. As a special case of fuzzy hypothesis testing, the p-value of crisp hypothesis as a fuzzy set on sample space, is called the fuzzy set description of the test. It's intercept indicates the rejection region of the test, and the level of the cut set is the significance level of the test. Finally, examples of discrete and continuous distributions are given to calculate the p-value of the fuzzy hypothesis.

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