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Numerical Solutions of Convective Diffusion Equations using Wavelet Collocation Method

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Abstract. Some partial differential equations appear in many application fields. Therefore, the discussion of numerical solutions of those partial differential equations using numerical methods becomes a valuable and important issue in numerical simulation. In numerical methods, the wavelet-collocation method has been frequently developed for solving PDEs, and the algorithm has yielded substantial results. However, theoretical research of the numerical solution has been rarely discussed yet. In this paper, the numerical solution of convective diffusion equations using the wavelet-collocation method is established, and its existence and uniqueness are derived.

Keywords: Numerical solutions; convective diffusion equations; wavelet collocation method.

1. Introduction

Some partial differential equations, such as convection diffusion equations [1], Helmholtz equations [2,3], appear in many application fields. For instance, convectio diffusion equations have been frequently used in electronic science, environmental science, and hydrodynamics, while Helmholtz equations have been widely used for analyzing acoustics, the vibration of membranes and other structures, wave scattering, electromagnetic fields. Consequently, the study of numerical solutions of those partial differential equations using numerical methods becomes a valuable and important issue in numerical simulation.

In numerical methods, the wavelet-collocation method has been frequently developed for solving PDEs, and the algorithm has yielded substantial results [4-6]. To our best knowledge, theoretical research of the numerical solution has been rarely discussed yet [7]. In order to improve the efficiency of existing wavelet collocation algorithms and develop more efficient wavelet collocation methods, some new wavelet-collocation methods can be constructed and introduced. Moreover, as the existence and uniqueness theory, it is the tool that leads us to infer that there exists only one numerical solution to PDEs [8]. Therefore, it is significant to develop the theoretical study of the numerical solution of PDEs.

2. Numerical solutions of convective diffusion equations

Here we consider the convective diffusion equation as follows:

$$\begin{cases} c(\mathbf{x})\dfrac{\partial v}{\partial t}+b(\mathbf{x})\cdot\nabla v-a(\mathbf{x})\nabla^2 v=\ddot{f},\mathbf{x}\in D, \quad t>0, \\ v(\mathbf{x},0)=v_0,\mathbf{x}\in D, \\ v(\mathbf{x},t)=\ddot{f}_0,\mathbf{x}\in\partial D,t>0, \end{cases} \tag{1}$$

where $a(x) \geq a0 \geq 0$, $\mathbf{x}=(x,y)^T$, ∂D is the boundary of the bounded region D.

Let $v(\mathbf{x})\approx v_j(\mathbf{x})$, we have an approximation as follows:

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$$v(\mathbf{x}) \approx v_j(\mathbf{x}) = \sum_{n_1=0}^{2^j}\sum_{n_2=0}^{2^j} \kappa_{j,n_1,n_2}\phi_{j,n_1,n_2}(\mathbf{x}-\mathbf{x}_m), \tag{2}$$

where $\kappa_{j,n_1,n_2} = v_j(x_{n_1}, y_{n_2})$, $\mathbf{x} = (x, y)^T$, $\mathbf{x}_m \in D \cup \partial D$, $\phi_{j,n_1,n_2}(\mathbf{x}) = \phi_{j,n_1,n_2}(x, y) = \phi_{j,n_1}(x)\phi_{j,n_2}(y)$ [7], and m = 0, 1, 2,$\cdots, 2^{j+2} + (2^j - 1)\times(2^j - 1)$.

By (1) and (2), we have

$$\begin{cases} v_j^{n+1} = \sum_{n_1=0}^{2^j}\sum_{n_2=0}^{2^j} \kappa_{j,n_1,n_2}\left(\phi_{j,n_1,n_2} - \frac{\Delta t b\nabla(\phi_{j,n_1,n_2})}{c} + \frac{\Delta t a\nabla^2(\phi_{j,n_1,n_2})}{c}\right) + \frac{\Delta t\ddot{f}^n}{c}, \\ \qquad m = 2^{j+2}+1, \cdots, 2^{j+2} + (2^j-1)\times(2^j-1)) \\ \sum_{n_1=0}^{2^j}\sum_{n_2=0}^{2^j} \kappa_{j,n_1,n_2}\phi_{j,n_1,n_2} = v_j^0, m = 2^{j+2}+1, \cdots, 2^{j+2} + (2^j-1)\times(2^j-1)), \\ \sum_{n_1=0}^{2^j}\sum_{n_2=0}^{2^j} \kappa_{j,n_1,n_2}\iota_{j,n_1,n_2} = \ddot{f}_0^n, \quad m = 0, 1, \cdots, 2^{j+2}, \end{cases} \tag{3}$$

where tn = nΔt, Δt > 0, and $\phi_{j,n_1,n_2} = \phi_{j,n_1,n_2}(\mathbf{x} - \mathbf{x}_m)$.

Now, we define the matrice as follows:

$$A = \begin{pmatrix} \phi_{j,n_1(x_0),n_2(x_0)}(\mathbf{x}_0 - \mathbf{x}_0) & \cdots & \phi_{j,n_1(x_0),2^{j+2}+(2^j-1)\times(2^j-1)}(\mathbf{x}_0 - \mathbf{x}_{2^{j+2}+(2^j-1)\times(2^j-1)}) \\ \vdots & \ddots & \vdots \\ \phi_{j,n_1(x_{2^{j+2}}),n_2(x_0)}(\mathbf{x}_{2^{j+2}} - \mathbf{x}_0) & \cdots & \phi_{j,n_1(x_{2^{j+2}}),2^{j+2}+(2^j-1)\times(2^j-1)}(\mathbf{x}_{2^{j+2}} - \mathbf{x}_{2^{j+2}+(2^j-1)\times(2^j-1)}) \\ \vdots & \ddots & \vdots \\ \phi_{j,2^{j+2}+(2^j-1)\times(2^j-1),x_0}(\mathbf{x}_{2^{j+2}+(2^j-1)\times(2^j-1)} - \mathbf{x}_0) & \cdots & \phi_{j,2^{j+2}+(2^j-1)\times(2^j-1),2^{j+2}+(2^j-1)\times(2^j-1)}(\mathbf{x}_{2^{j+2}+(2^j-1)\times(2^j-1)} - \mathbf{x}_{2^{j+2}+(2^j-1)\times(2^j-1)}) \end{pmatrix}$$

and let:

$\mathrm{K} = (\kappa_0^n, \cdots, \kappa_{2^{j+2}}^n, \kappa_{2^{j+2}+1}^n, \cdots, \kappa_{2^{j+2}+(2^j-1)\times(2^j-1)}^n)^T$, $\Upsilon = (\ddot{f}_0^n(\mathbf{x}_0), \cdots, \ddot{f}_0^n(\mathbf{x}_{2^{j+2}}), v_j^{n+1}(\mathbf{x}_{2^{j+2}+1}), \cdots, v_j^{n+1}(\mathbf{x}_{2^{j+2}+(2^j-1)\times(2^j-1)}))^T$.

Then, at time t = tn, (3) can be expressed a matrix equation as follows:

$$A\mathrm{K} = \Upsilon \tag{4}$$

Regarding (4), we have the theorem as follows:

Theorem. If $F[\phi_j]$ is almost everywhere larger than 0, the matrix equation $A\mathrm{K} = \Upsilon$ has a unique numerical solution, where the basis function ϕ_j is symmetrical, and $F[\phi_j]$ is Fourier transform.

Proof. First, we derive that $A = \{\phi_{j,m}(\mathbf{x}_k)\}_{N\times N}$ is symmetrical, because by the symmetry of ϕ_j, where, $N\times N = (2^{j+2} + (2^j-1)\times(2^j-1))\times(2^{j+2} + (2^j-1)\times(2^j-1))$.

Now, we show $A = \{\phi_{j,m}(\mathbf{x}_k)\}_{N\times N}$ is definite. $\forall v \neq 0 \in R^N$,

$$\begin{aligned}(Av,\ v) &= \sum_{m=0}^{N}\sum_{k=0}^{N} v_m v_k \phi_{j,m}(\mathbf{x}_k) = \sum_{m=0}^{N}\sum_{k=0}^{N} v_m v_k \phi_j(\mathbf{x}_k - \mathbf{x}_m) \\ &= \sum_{m=0}^{N}\sum_{k=0}^{N} v_m v_k (\frac{1}{2\pi})^n \int_{-\infty}^{+\infty} F[\phi_j] e^{i<\omega,\mathbf{x}_k - \mathbf{x}_m>} d\omega \\ &= (\frac{1}{2\pi})^n \int_{-\infty}^{+\infty} F[\phi_j] \sum_{k=0}^{N} v_k e^{i<\omega,\mathbf{x}_k>} \sum_{m=0}^{N} v_j e^{-i<\omega,\mathbf{x}_m>} d\omega \\ &= (\frac{1}{2\pi})^n \int_{-\infty}^{+\infty} F[\phi_j] |v(\omega)|^2 d\omega > 0 \end{aligned} \quad , (5)$$

where $F[\phi_j]$ is almost everywhere larger than 0, and $v(\omega) = \sum_{m=0}^{N} v_m e^{i<\omega,\mathbf{x}_m>}$.

Thus A is definite. Hence, $|A| \neq 0$.

Now, substitute K into Equation (2), we can achieve a numerical solution of (1) at time t = tn.

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3. Some examples and numerical analysis

Example 1

Consider 1-dimensional problems:

$$\begin{cases} \frac{\partial v(x,t)}{\partial t}+\frac{\partial v(x,t)}{\partial x}-\frac{\partial^2 v(x,t)}{\partial x^2}=f,x\in D, \\ v(x,0)=v_0,x\in D, \\ v(0,t)=g_1, \\ v(2,t)=g_N, \end{cases} \tag{5}$$

where $v(x,t)=e^{10^{-3}t}x(1-x)$, which is the analytical solution of (6). t > 0, and D = [0,2]. f, v_0, g_1 and g_N are satisfied by the analytical solution. ϕ_j is the Quasi-Shannon scaling function. The error estimation of the algorithm is carried out with L2-norm, the space step h = 1/2j, and Δt = 0.01. The numerical results of the wavelet-collocation algorithm are shown in Table 1.

Table 1. The numerical results of(6)

t	Calculation Error
t = 0.3	5.8163e-004
t = 0.6	1.9291e-004
t = 1.2	3.7309e-004
t = 8	4.7321e-004

From calculation error, it is seen that wavelet-collocation method for 1-dimensional equations is feasible, and the algorithm achieves satisfactory convergence results.

Simultaneously, the finite element method (FEM) [7] is used for solving Example 1, and the numerical solutions of the FEM are shown in Fig. 3.2.

First, from Fig.3.1-Fig.3.2 it is seen that the algorithm of the paper achieves satisfactory results, while the FEM has similar convergent results. Second, we summarize the analysis indexes of the two algorithms regarding calculation error and computation time as follows:

Table 2. Comparison of the numerical results

t	Relative Conditions	The wavelet-collocation method	FEM
0.5	Calculation Error	1.5710e-004	4.2299e-004
	Computation Time (*s*)	0.407000	0.422000
2	Calculation Error	1.6011e-004	4.5139e-004
	Computation Time (*s*)	1.109000	1.203000
10	Calculation Error	1.6021e-004	4.5224e-004
	Computation Time (*s*)	5.2030000	6.047000

In Table 2, it is seen that the wavelet-collocation method requires less computation time, and it produces less error. Therefore, when compared to the FEM, the wavelet-collocation method achieves much better results.

Example 2

Consider 2-dimensional problems:

$$\begin{cases} \frac{\partial v}{\partial t}+\nabla v-\nabla^2 v=f,(x,y)\in D, \\ v(x,y,0)=v_0,(x,y)\in D, \\ v(x,y,t)=g,(x,y)\in \partial D, \end{cases} \tag{6}$$

where $v(x,y,t)=t^2\sin(\pi x)\sin(\pi y)$, which is the analytical solution of (7). t > 0, and D = [0,1]×[0,1]. f, v_0 and g are satisfied by the analytical solution. ϕ_j is the Quasi-Shannon scaling

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function. The error estimation of the algorithm is carried out with L2-norm, the space step h = 1/2j, and Δt = 0.01. The numerical results of the algorithm are shown in Figure 3.3- Figure 3.6 and Table 3.

Table 3. The numerical results of (7)

t	*Calculation Error*	*t*	*Calculation Error*
t = 0.2	1.7582e-003	*t*=0.25	1.6160e-004
t = 0.6	1.4433e-003	*t* = 0.7	2.9568e-004
t =1.2	1.2832e-003	*t* =1.3	2.2334e-004
t = 6	1.7305e-003	*t* = 6.5	1.4311e-004

From Figure 3-Figure 6 and Table 3, it is seen the wavelet-collocation method for 2-dimensional equations is feasible, and the algorithm achieves much higher accuracy.

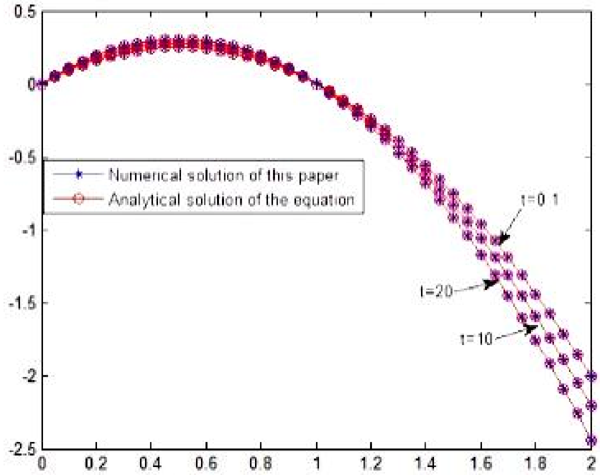

Fig 1 Numerical solution of this

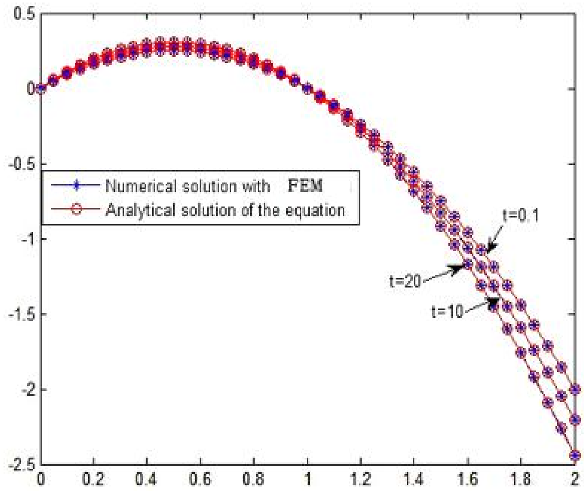

Fig 2 Numerical solution of FEM

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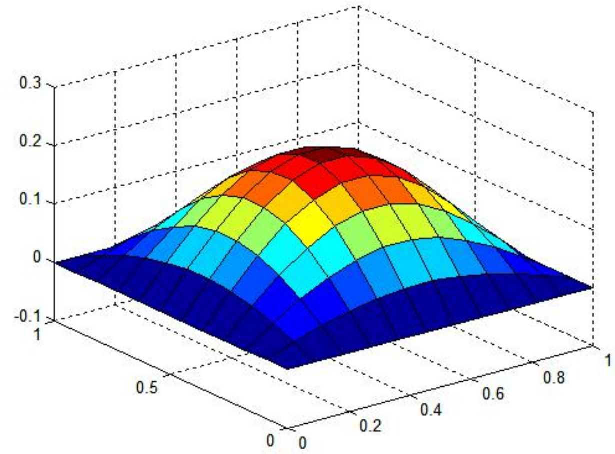

Fig 3 Numerical solution ($t = 0.2$)

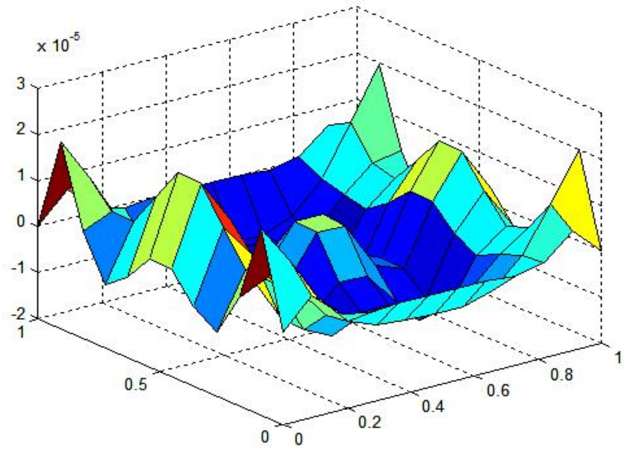

Fig 4 Calculation error ($t = 0.2$)

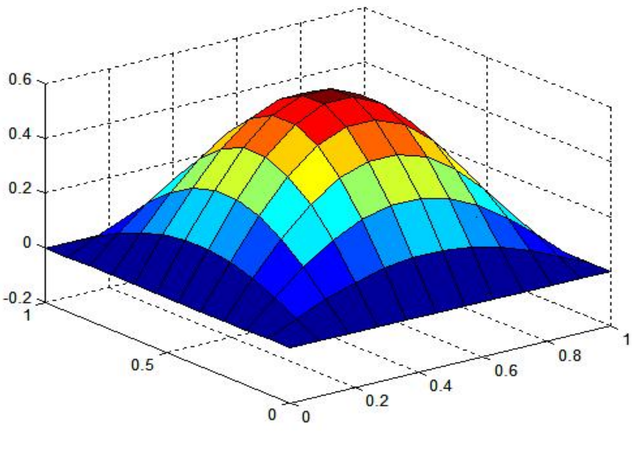

Fig 5 Numerical solution ($t = 0.6$)

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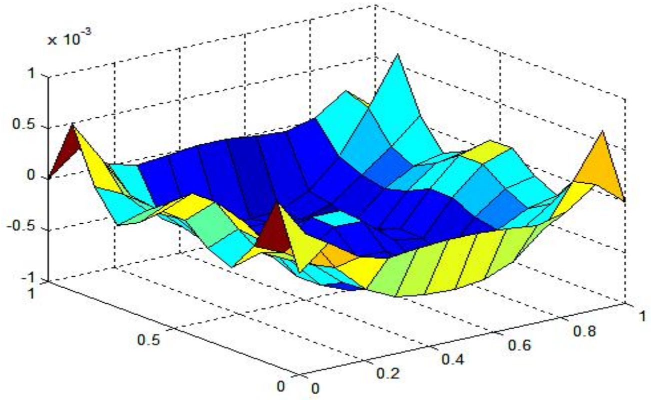

Fig 6 Calculation error (t = 0.6)

4. Conclusions

The existence and uniqueness theory of the numerical solution of convective diffusion equations is established and discussed. By the theory in this paper, it is easy to verify the solvability of convective diffusion equations using the wavelet-collocation method. Therefore, the discussion of the numerical solution is valuable for developing wavelet-collocation methods.

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