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Triangular Norm based Invex Fuzzy Sets

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Abstract. Zadeh's convex fuzzy set (CFS for short) have two important properties, (1) fuzzy set (FS for short) is convex iff its cuts are convex, (2) arbitrary intersection of CFSs is a CFS. Generalized CFS, named-invex FS is introduced. Meanwhile, two properties above are generalized into-invex FS.

Keywords: CFS, -norm, invex set, -invex FS.

1. Introduction

Convex sets and convex functions attaches importance to mathematical economics, mathematical programming, management science, engineering and other subjects. As is known to us, a subset X of $\mathbb{R}^n$ is a convex set if for all $p,q \in X$ and $\lambda \in [0,1]$

$$p,q \in X \Rightarrow \lambda p + (1-\lambda)q \in X. \tag{1}$$

In inequality constrained optimization, in order to expand the application of convex sets, many generalized convex sets are proposed. One of these generalized convexity is invex set.Invex set was introduced by Hanson [4]. Let $X \subseteq \mathbb{R}^n$, X is invex w. r.t $\omega : X \times X \to X$, if for all $\lambda \in [0,1]$, it can be summarized that

$$p,q \in X \Rightarrow q + \lambda\omega(p,q) \in X. \tag{2}$$

In resent years, convex functions and invex functions have been generalized to fuzzy case by many scholars , for example, authors in [1], [2], [4], [6], [10-13], [17], [18], [20], [26] have studied generalized convex fuzzy mapping According to these results, fuzzy convex functions and fuzzy invex functions are $f : \mathbb{R}^n \to E$, in which E denotes the set of fuzzy numbers, i.e. there exists fuzzy convex functions and fuzzy invex functions are the functions which are from crisp sets $\mathbb{R}^n$ to FSs E. How do we generalize convex sets to fuzzy case?

Zadeh's CFS [25], which is generalized the convex subset as follow: Let $X \subseteq \mathbb{R}^n$, a FS $\tilde{C} : X \to [0,1]$ is convex if for all $p,q \in X$ and $\lambda \in [0,1]$, it can be summarized that

$$\tilde{C}(\lambda p + (1-\lambda)q) \ge \min\{\tilde{C}(p), \tilde{C}(q)\}. \tag{3}$$

Properties of generalized fuzzy convexity were considered by many authors ([5], [14-16], [21], [23], [24]). $(t,\mu]$-CFS is one of generalized the notion of Zadeh's CFS, which is proposed by Yuan and Lee [24]. Let $t,\mu \in (0,1]$, and $t < \mu$, A FS $\tilde{C} : X \to [0,1]$ is called a $(t,\mu]$ - CFS if for any $p,q \in X$ and $\lambda \in [0,1]$

$$\tilde{C}(\lambda p + (1-\lambda)q) \vee t \ge \min\{\tilde{C}(p), \tilde{C}(q), \mu\}. \tag{4}$$

Yuan and Lee[24] also proposed another generalizations of fuzzy convexity called R - CFS as: Let R be a operator (we use → instead of R implication in this paper) over $[0,1]$ and $t \in [0,1]$. A FS $\tilde{C} : X \to [0,1]$ is called R - CFS if for any $p,q \in X$ and $\lambda \in [0,1]$ and therefore

$$\min\{\tilde{C}(p),\ \tilde{C}(q)\} \to \tilde{C}(\lambda p + (1-\lambda)q)) \ge t. \tag{5}$$

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Pan [15] proposed graded CFS based on triangular norms which is more general than $(t,\mu]$-CFS and R- CFS. A fuzzy set $\tilde{C}: X \to [0,1]$ is said to be $(t,\otimes)$-CFS ($\otimes$ is a triangular norm and $t \in [0,1]$ if for all $p,q \in X$ and $\lambda \in [0,1]$, it can be summarized that

$$\tilde{C}(\lambda p + (1-\lambda)q)) \geq \tilde{C}(p) \otimes \tilde{C}(q) \otimes t. \tag{6}$$

In this paper, we continue to study generalized CFS. The paper is organized as: It reviews fundamental notions and properties of t-norm and fuzzy implications. By employing t-norm, it proposes a kind of triangular norm based invex FS $(t,\otimes)$ -invex fuzzy set) and discuss some properties in Sect.2. Sect.3 is conclusion.

2. Triangular norm based invex fuzzy sets

(Fuzzy) implication function is a function $[0,1]\times[0,1]\to[0,1]$. Typically, any fuzzy concept of implication must generalize the corresponding crisp concept and required that $0 \to 1 = 0 \to 0 = 1 \to 1 = 1$ and $1 \to 0 = 0$.

Definition 2.1:([7],[8]) A t-norm is anoperation $\otimes:[0,1]\times[0,1]\to[0,1]$, conforms to increasing, associative, commutative and

$1 \otimes a = a, \forall a \in [0,1].$

Definition2.2:([3],[9]) An R-implication (or residuated implication) is defined by

$$p \to_{\otimes} q = \sup\{r \in [0,1] | \, p \otimes r \leq q\}, p,q \in [0,1], \tag{7}$$

where $\otimes$ is a t-norm on $[0, 1]$.

Example2.1: The most important t-norms, which are used here, named Łukasiewicz and Godel implications (see [7], [22] for their definitions), which are given together with the corresponding R implication operators as following:

$$p \to_{Lu} q = (1 - p + q) \wedge 1,$$
$$p \otimes_{Lu} q = (p + q - 1) \vee 0.$$
$$p \to_G q = \begin{cases} 1, p \leq q \\ q, p > q \end{cases},$$
$$p \otimes_G q = p \wedge q.$$

Spontaneously, we would like to make the notion of suppprequasiincave FS generalized, from view point of t-norm based fuzzy logic, And the notion of $(t,\otimes)$ -invex FS is introduced. In this section, $\otimes$ always mean a t-norm, $\to_{\otimes}$ is the corresponding R implication; t always mean a number in $(0, 1]$. For each such FS $\tilde{C}$ on $\mathbb{R}^n$ (i.e. FS is a mapping $\tilde{C}:\mathbb{R}^n \to [0,1]$), we write $\tilde{C}_\alpha = \{x \in \mathbb{R}^n : \tilde{C}(x) \geq \alpha\}$ $(\forall \alpha \in (0,1])$ (named α -cut of $\tilde{C}$) and $Supp\tilde{C} = \{x \in \mathbb{R}^n : \tilde{C}(x) > 0\}$ (named support of $\tilde{C}$).

Definition 2.3: Let $X \subseteq \mathbb{R}^n$, a FS $\tilde{C}$ is said to be $(t,\otimes)$ -invex FS w. r.t $\omega: X \times X \to X$ if for any $p,q \in X$ and $\lambda \in [0,1]$, it can be summarized that

$$\tilde{C}(q + \lambda\omega(p,q)) \geq \tilde{C}(p) \otimes \tilde{C}(q) \otimes t. \tag{8}$$

Remark 2.1: (1) From (8), we can observe that, any FS $\tilde{C}$ is $(t,\otimes)$ -invex FS w.r.t $\omega(p,q) = 0$, this fact can also be seen in invex set (see (3)). Because main purpose of invex set is to define invex function, we often consider $\omega(p,q) \neq 0$ in (3) and (8).

(2)If $\otimes$ is Godel t-norm, any $(1,\otimes)$ -invex FS is a Syau's [19] supp-prequasiincave FS. If $\omega(p,q) = p - q$, any $(t,\otimes)$ -invex FS is a Pan's [15] $(t,\otimes)$ -CFS. If $\omega(p,q) = p - q$ and t=1, any $(t,\otimes)$ -invex FS is a Yuan's[24] $\otimes$- CFS. If $\omega(p,q) = p - q$ and t=1, and $\otimes$ is Godel t-norm , any $(t,\otimes)$ -invex FS is a Zadeh′ s CFS can be seen obviously.

The advantages of $(t,\otimes)$ -invex FS are shown in the example below, respect to Syau 's [19] supp-prequasiincave FS and Pan's[15] $(t,\otimes)$ -CFS.

Example 2.2:Let

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$$\tilde{C}(p)=\begin{cases}0.9, & p\in(3,5]\cup[7,9];\\ 0.7, & p\in[2,3];\\ 0, & otherwise.\end{cases}$$

$$\omega(p,q)=\begin{cases}p-q, & p\geq 7,q\geq 7;\\ p-q, & p<5,q<5;\\ 2-q, & p\geq 7,q\leq 5;\\ 7-q, & p\leq 5,q>7.\end{cases}$$

Then, $\tilde{C}$ is a $(t,\otimes_G)$ -invex FS on $\mathbb{R}$ for any $t\in(0,0.7]$, Also $\tilde{C}$ is a $(t,\otimes_{Lu})$ -invex FS on $\mathbb{R}$ for any $t\in(0,0.9]$. However,

Let $p=8,q=5,\lambda=1$,then

$\tilde{C}(\lambda p+(1-\lambda)q)=\tilde{C}(2)=0.7<0.9=\min\{\tilde{C}(8),\tilde{C}(5)\}$.

Thus, $\tilde{C}$ is not a supp-prequasiincave FS on $\mathbb{R}$ w.r.t $\omega(p,q)$ on $\mathbb{R}$.

(2) If we let $p=8,q=4,\lambda=\frac{1}{2}$

$\tilde{C}(\lambda p+(1-\lambda)q)=\tilde{C}(6)=0.$

Then $\tilde{C}$ is not a $(t,\otimes)$ -invex FS on $\mathbb{R}$.

Proposition 2.1: $\tilde{C}$ is a $(t,\otimes)$ -invex FS w.r.t $\omega(p,q)$ iff

$$\tilde{C}(p)\to_{\otimes}(\tilde{C}(q)\to_{\otimes}\tilde{C}(q+\lambda\omega(p,q)))\geq t. \tag{9}$$

proof: The proof can be gained from Definition 2.2 and Definition 2.3.

Proposition 2.2: make $X\subseteq\mathbb{R}^n$ and $\tilde{C}$ become a FS on X. If $\tilde{C}_\alpha$ is an invex set w.r.t $\omega(p,q)$ for any $\alpha\in(0,1]$, then $\tilde{C}$ is a $(t,\otimes)$ -invex FS.

proof: Suppose $\tilde{C}_\alpha$ is an invex set w.r.t $\omega(p,q)$ for any $\alpha\in(0,1]$, then

$\tilde{C}(q+\lambda\omega(p,q))\geq\min\{\tilde{C}(p),\tilde{C}(q)\},$

i.e.

$\min\{\tilde{C}(p),\tilde{C}(q)\}\to_{\otimes}\tilde{C}(q+\lambda\omega(p,q))=1\geq t.$

Since

$\tilde{C}(p)\otimes\tilde{C}(q)\leq\min\{\tilde{C}(p),\tilde{C}(q)\},$

Thus

$\min\{\tilde{C}(p),\tilde{C}(q)\}\to_{\otimes}\tilde{C}(q+\lambda\omega(p,q))=1\geq t.$

Proposition 2.3: Let $X\subseteq\mathbb{R}^n$ and $\tilde{C}$ be a FS on X. Two equal conditions are listed below.

(1) $\tilde{C}$ is a $(t,\otimes_G)$ -invex FS w.r.t $\omega(p,q)$ on X;

(2) α -cuts of $\tilde{C}$ are invex sets w. r.t $\omega(p,q)$ for any $\alpha\leq t$.

proof: Let $\tilde{C}$ be a $(t,\otimes_G)$ -invexFS. Assume $p,q\in\tilde{C}_\alpha$, i.e., $\tilde{C}(p)\geq\alpha,\tilde{C}(q)\geq\alpha$.

Since $\tilde{C}$ is $(t,\otimes_G)$ -invex FS w.r.t $\omega(p,q)$ on X, then

$\tilde{C}(q+\lambda\omega(p,q))\geq\tilde{C}(p)\otimes_G\tilde{C}(q)\otimes_G t\geq\alpha\otimes_G\alpha\otimes_G\alpha=\alpha.$ thus, $q+\lambda\omega(p,q)\in\tilde{C}_\alpha.$

Let $\tilde{C}_\alpha$ are invex sets w.r.t $\omega(p,q)$ for any $\alpha\leq t$. Let $\alpha=\tilde{C}(p)\leq\tilde{C}(q)$, if $\alpha\leq t$, then $q\in\tilde{C}_\alpha$ and $\tilde{C}_\alpha$ is a invex set w.r.t $\omega(p,q)$, thus

$\tilde{C}(q+\lambda\omega(p,q))\geq\alpha=\tilde{C}(p)=\tilde{C}(p)\otimes_G\tilde{C}(q)\otimes_G t.$

If $\alpha>t$, then $\tilde{C}(p)\otimes_G\tilde{C}(q)\otimes_G t=t$. Because $\tilde{C}_\alpha$ is a invex set w.r.t $\omega(p,q)$, then $\alpha=\tilde{C}(p)\leq\tilde{C}(q)$ implies $\tilde{C}(q+\lambda\omega(p,q))\geq\alpha$, i.e.,

$\tilde{C}(q+\lambda\omega(p,q))\geq\alpha>t=\tilde{C}(p)\otimes_G\tilde{C}(q)\otimes_G t.$

If other t -norms are taken rather than $\otimes_G$ in $(t,\otimes_G)$ -CFS, then the result mentioned before is not always correct. Counterexample is given by Pan [15] when $\omega(p,q)=p-q$.

Proposition 2.4: Let $X\subseteq\mathbb{R}^n$ and $\tilde{C}_1$, $\tilde{C}_2$ be two $(t,\otimes)$ -invex FS w.r.t $\omega(p,q)$ on X, then $\tilde{C}_1\otimes\tilde{C}_2$ is $(t\otimes t,\otimes)$ -invex FS w.r.t $\omega(p,q)$ on X.

proof: As $\tilde{C}_1$ and $\tilde{C}_2$ are $(t,\otimes)$ -invex FS w. r.t $\omega(p,q)$ on X, for any $p,q\in X$ and $\lambda\in[0,1]$, from (8) we have

$(\tilde{C}_1\otimes\tilde{C}_2)(q+\lambda\omega(p,q))$

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$=\tilde{C}_1(q+\lambda\omega(p,q))\otimes\tilde{C}_2(q+\lambda\omega(p,q))$

$\geq(\tilde{C}_1(p)\otimes\tilde{C}_1(q)\otimes t)\otimes(\tilde{C}_2(p)\otimes\tilde{C}_2(q)\otimes t)$

$=(\tilde{C}_1(p)\otimes\tilde{C}_2(p))\otimes(\tilde{C}_1(q))\otimes\tilde{C}_2(q))\otimes(t\otimes t)$

$=(\tilde{C}_1(p)\otimes\tilde{C}_2(p))\otimes(\tilde{C}_1(q))\otimes\tilde{C}_2(q))\otimes(t\otimes t)$

$=(\tilde{C}_1\otimes\tilde{C}_2)(p)\otimes(\tilde{C}_1\otimes\tilde{C}_2)(q)\otimes(t\otimes t)$

Thus, $\tilde{C}_1\otimes\tilde{C}_2$ is a $(t\otimes t,\otimes)$ invex FS w.r.t $\omega(p,q)$ on X.

Due to the defined meaning of $\otimes_G$, we have:

Corollary 2.1: Let $X\subseteq\mathbb{R}^n$ and $\tilde{C}_1$ $\tilde{C}_2$ be $(t,\otimes)$ -invex FS w. r.t $\omega(p,q)$ on X, then $\tilde{C}_1\cap\tilde{C}_2$ is a $(t,\otimes_G)$ -invex FS w.r.t $\omega(p,q)$ on X.

Example 2.3: the next one to 2.2

(1) Since $\tilde{C}_1(p)=\begin{cases}0.9, & p\in(3,5]\cup[7,9];\\ 0.7, & p\in[2,3];\\ 0, & otherwise.\end{cases}$ and $\tilde{C}_2(p)=\begin{cases}0.8, & p\in(3,5]\cup[7,9];\\ 0.7, & p\in[2,3];\\ 0.2, & otherwise.\end{cases}$ are $(0.7,\otimes_G)$ -invex FS w.r.t $\omega(p,q)=\begin{cases}p-q, & p\geq 7,q\geq 7;\\ p-q, & p<5,q<5;\\ 2-q, & p\geq 7,q\leq 5;\\ 7-q, & p\leq 5,q>7.\end{cases}$ on $\mathbb{R}$, from Corollary 2.1,

we have $(\tilde{C}_1\cap\tilde{C}_2)(p)=\begin{cases}0.8, & p\in(3,5]\cup[7,9];\\ 0.7, & p\in[2,3];\\ 0, & otherwise.\end{cases}$ is a $(0.7,\otimes_G)$ -invex FS w.r.t $\omega(p,q)$ on $\mathbb{R}$.

(2) Since $\tilde{C}_1(p)=\begin{cases}0.9, & p\in(3,5]\cup[7,9];\\ 0.7, & p\in[2,3];\\ 0, & otherwise.\end{cases}$ and $\tilde{C}_2(p)=\begin{cases}0.8, & p\in(3,5]\cup[7,9];\\ 0.7, & p\in[2,3];\\ 0.2, & otherwise.\end{cases}$ are $(0.8,\otimes_{Lu})$ -invex FS w. r.t $\omega(p,q)$ on $\mathbb{R}$, from Proposition 2.4, we have $(\tilde{C}_1\otimes_{Lu}\tilde{C}_2)(p)=\begin{cases}0.7, & p\in(3,5]\cup[7,9];\\ 0.4, & p\in[2,3];\\ 0, & otherwise.\end{cases}$ is a $(0.6,\otimes_{Lu})$ -invex FSw. r.t $\omega(p,q)$ on $\mathbb{R}$.

3. Conclusions

Generalized CFSs has been delved into in the literature by many professional scholars. With the aim of introducing the concept of $(t,\otimes)$ -invex fuzzy set, the relationship between this generalized CFSs and their cuts and other properties are also discussed.

References

[1] M. Arana-Jimenez, A. Rufian-Lizana, Y. Chalco-Cano, H. Roman-Flores, "Generalized convexity in fuzzy vector optimization through a linear ordering," Inf. Sci., vol.312, pp.13-24, 2015.

[2] B. Farhadinia, " Necessary optimality conditions for fuzzy variational problems," Inf. Sci., vol.181,pp.1348-1357, 2011.

[3] J.Fodor, M.Roubens, Fuzzy preference modelling and multicriteria decision support. Theory and Decision Library. Kluwer Academic Publishers, Dordrecht/Boston/ London, 1994.

[4] M.A. Hanson, "On sufficiency of the Kuhn-Tucker conditions," J. Math.Anal. Appl., vol.80, pp.545-550,1981.

[5] X.J. Hua, X.L. Xin, X.Zhu, "Generalized convex fuzzy sublattices," Comput. Math. Appl.,vol.62,pp.699-708,2011.

[6] M. Inuiguchi, "Necessity measure optimization in linear programmingproblems with fuzzy polytopes," Fuzzy Sets Syst., vol.158, pp.1882-1891, 2007 .

[7] P.Hajek, Metamethematics of Fuzzy Logic, Dordrecht: Kluwer Academic Publishers, 1998.

ISSN:2790-1688 DOI: 10.56028/aetr.1.1.198
[8] E. P.Klement, R. Mesiar, E. Pap, Triangular norms. Dordrecht: Kluwer.ISBN 0-7923-6416-3, 2000.

[9] G.J. Klir, B.Yuan, Fuzzy sets and fuzzy logic. Theory and Applications. Prentice-Hall, PTR, NJ.1995.

[10] L.Stefanini, M. Arana-Jimenez, "Karush-Kuhn-Tucker conditions forinterval and fuzzy optimization in several variables under total and directional generalized differentiability, " Fuzzy Sets Syst., vol.362, pp.1-34, 2019.

[11] L. F.Li, S.Y.Liu, J.K.Zhang, "On interval-valued invex mappings and optimality conditions for interval-valued opti-mization problems," Journal of Inequalities and Applications, vol.1, pp.1-19, 2015.

[12] L. F.Li, S.Y.Liu, J.K.Zhang, "On fuzzy generalized convex mappings and optimality conditions for fuzzy weakly univex mappings," Fuzzy Sets Syst., vol.280, pp. 107-132, 2015.

[13] S.Nanda, K.Kar, " Convex fuzzy mappings," Fuzzy Sets Syst., vol.32, pp.359-367, 1989.

[14] K. Nourouzi, A. Aghajani, "Convexity in triangular norm of fuzzy sets," Chaos Solitons Fractals, vol. 36, pp.883-889, 2008.

[15] X.D. Pan, D. Meng, "Triangular norm based graded convex fuzzy sets," Fuzzy Sets Syst., vol.209,pp.1-13, 2012 .

[16] S. Jenei, "Amore efficient method for defining fuzzy connectives," Fuzzy Sets Syst., vol. 90, pp.25-35, 1997.

[17] R.Osuna-Gomez, Y.Chalco-Cano, B.Hernandez-Jimenez, I.Aguirre-Cipe, "Optimality conditions for fuzzy constrained programming problems," Fuzzy Sets Syst., vol. 362,pp.35-54, 2019.

[18] J.Ramik, M.Vlach, Generalized concavity in fuzzy optimization and decision analysis, Kluwer Academics Pub- lishers, Dordrecht, Boston, London, 2002.

[19] Y.R. Syau, E. "Stanley Lee, Preincavity and fuzzy decision making," Fuzzy Sets Syst., vol.155, pp. 408-424, 2005 .

[20] Y.R.Syau, "Invex and generalized convex fuzzy mappings," Fuzzy Sets Syst., vol. 115, pp.455-461, 2000.

[21] H. Tahayori, G. B. Tettamanzi, G. D. Antoni, Visconti, "A.: On the calculation of extended max and min operations between convex fuzzy sets of the real line," Fuzzy Sets Syst., vol.160, pp. 3103-3114, 2009.

[22] G.J.Wang, Theory of Non-classic Mathematical Logic and Approximate Reasoning, Beijing: Science in China Press, 2000(in chinese).

[23] X. M. Yang, "A note on convex fuzzy sets," Fuzzy Sets Syst., vol.53, pp.117-118, 1993.

[24] X.H.Yuan, E. S. Lee, "The definition of convex fuzzy subset," Comput.Math. Appl., vol. 47, pp.101-113, 2004.

[25] L.A. Zadeh, "Fuzzy sets," Information and Control, vol.8 pp.338-353, 1965.

[26] J.K.Zhang, et al., "The KKT optimality conditions in a class of gener-alized convex optimization problems with an interval-valued objective function," Optim Lett., vol.8, pp.607-631, 2014.