ISSN:2790-1688 DOI: 10.56028/aetr.1.1.94

On Network Engergy of Random Graphs

Shengjiu Liu[1,a], Minghui Ou[2,b]

[1] Big Data and Internet of Things School,Chongqing Vocational Institute of Engineering, Chongqing, China

[2] Big Data and Internet of Things School,Chongqing Vocational Institute of Engineering, Chongqing, China

[a] e-mail: liushengjiu2008@163.com, [b] e-mail: omhcumt_1992@163.com

Abstract. In this paper, the network energy of undirected graph, oriented graph, and mixed graph on random graphs will be studied. We compute energy and network energy on random undirected graphs, skew energy and network energy on random oriented graphs, and Hermitian energy and network energy on random mixed graphs by upper bounds of them with theoretical methods. By comparing network energy and other energies of undirected graph, oriented graph, and mixed graph of random graphs, we obtain some relations among network energy and other energies

Keywords: network energy; energy; skew energy; Hermitian energy; undirected graph; oriented graph; mixed graph; random graph.

1. Introduction

Let G={V(G), E(G)} be a simple undirected graph with vertex set V(G) and edge set E(G), while E(G)⊂V(G) ×V(G)\{{u, u}: u∈V(G)}. If $E(G) = m$, at the same time, $V(G) = n$, the undirected graph G can be denoted as (n, m)-undirected graph, named by G(n, m). $NE(G)$, the network energy of G(n, m), can be defined as follows [1]:

$$NE(G) = n^{\sqrt{log_n 2m}} \tag{1}$$

Let Gσ={V(Gσ), E(Gσ)} be an oriented graph, and σ, the orientation on the edge set E(G), the oriented graph Gσ is called an (n, m)-oriented graph, denoted by Gσ(n, m). The network energy NE(Gσ) of Gσ(n, m) is defined as [2]:

$$NE(G^{\sigma}) = n^{\sqrt{log_n m}} \tag{2}$$

Let Gϕ={V(Gϕ), E(Gϕ)} be a mixed graph, and σ, the orientation on a subset of the edge set E(G). The network energy NE(Gϕ) of Gϕ with $\overline{m}$ undirected edges and $\vec{m}$ directed edges is defined as [3]:

$$NE(G^{\phi}) = n^{\sqrt{log_n \left(2\overline{m}+\vec{m}\right)}} \tag{3}$$

Keep in mind that:

$$\overline{m} + \vec{m} = m \tag{4}$$

If $\overline{m}$ and $\vec{m}$ satisfy the following equation:

$$\frac{\vec{m}}{\overline{m} + \vec{m}} = r \tag{5}$$

Joint with equations (3)-(5), the following equation can be obtained:

$$NE(G^{\phi}) = n^{\sqrt{log_n (2-r)m}} \tag{6}$$

In this paper, the network energy of undirected graph, oriented graph, and mixed graph on random graphs will be studied. We compute network energy with other energies, e.g., energy, skew energy, Hermitian energy, etc, of random undirected graphs, random oriented graphs, and random mixed graphs. We obtain some relations among network energy and others energies.

ISSN:2790-1688 DOI: 10.56028/aetr.1.1.94

The rest of the paper is organized as follows. In Section 2, we present energies of undirected graph, oriented graph, and mixed graph, e.g., energy skew energy, and Hermitian energy, and present the bounds of network energy, energy, skew energy, Hermitian energy of random graphs. In Section 3, we show relations among network energies of undirected graph, oriented graph, and mixed graph. In Section 4, we get relations among energy, skew energy, and Hermitian energy of undirected graph, oriented graph, and mixed graph of random graphs respectively. Relations among network energy, energy, skew energy, and Hermitian energy of random graphs, including undirected graph, oriented graph, and mixed graph, etc, are obtained in Section 5. The conclusion of this paper is shown in Section 6 at last.

2. Preliminaries

2.1 Energy , Skew energy, and Hermitian energy

Let A(G) be the adjacency matrix of G={V(G), E(G)}. Hence, if the vertex vi and the vertex vj are adjacent, thus, Aij=Aji = 1, otherwise, Aij=Aji=0, vi, vj∈V(G). E(G), the energy of G, can be shown as follows [4]:

$$E(G) = \sum_{i=1}^{n} \left|\lambda_i A(G)\right| \tag{7}$$

where λ1A(G), λ2A(G) , ..., λiA(G), ... , λnA(G) are denoted of the eigenvalues of A(G).

Let S(Gσ) is the skew adjacency matrix of Gσ={V(Gσ), E(Gσ)}. Hence, if $\overrightarrow{v_i v_j}$ is a directed edge of Gσ, thus, Sij=-Sji = 1, otherwise, Sij=Sji=0, vi, vj∈V(Gσ). εS(Gσ), the skew energy of Gσ, can be shown as follows [5]:

$$\varepsilon_S(G^\sigma) = \sum_{i=1}^{n} \left|\lambda_i S(G^\sigma)\right| \tag{8}$$

where λ1S(Gσ), λ2S(Gσ) , ..., λiS(Gσ), ... , λnS(Gσ) are denoted of the eigenvalues of S(Gσ).

Let H(Gϕ) is the Hermitian adjacency matrix of Gϕ={V(Gϕ), E(Gϕ)}. Hence, if $\overrightarrow{v_i v_j}$ is a directed edge of Gϕ, thus, Hij=-Hji = i, while if $\overline{v_i v_j}$ is an undirected edge of Gϕ, thus, Hij=Hji = 1, otherwise, Sij=Sji=0, vi, vj∈V(Gσ). εH(Gϕ), the Hermitian energy of Gϕ, can be shown as follows [6]:

$$\varepsilon_H(G^\phi) = \sum_{i=1}^{n} \left|\lambda_i H(G^\phi)\right| \tag{9}$$

where λ1H(Gϕ), λ2H(Gϕ) , ..., λiH(Gϕ), ... , λnH(Gϕ) are denoted of the eigenvalues of H(Gσ).

2.2 Bounds of Random Graph

Let Gn(p) be a random graph with n vertices. What's more, all the edges in the graph Gn(p) are connected with p one by one. In this paper, all p is a real constant with p<1, and at the same time, p>0, for convenience of our description [7].

Theorem 2.1 Let G be a simple, undirected, and a finite graph, while a random graph Gn(p), at the same time. Thus, NE(Gn(p)), the network energy of G, enjoys the inequality as follows [1]:

$$NE(G) < n^{3/2}\sqrt{p} \tag{10}$$

Theorem 2.2 Let Gσ be a simple, oriented, and a finite graph, while G, the underlying graph is a random graph Gn(p), at the same time. Thus, NE(Gσ), the network energy of Gσ, enjoys the inequality as follows [2]:

$$NE(G^\sigma) < \frac{\sqrt{2}}{2} n^{3/2}\sqrt{p} \tag{11}$$

ISSN:2790-1688 DOI: 10.56028/aetr.1.1.94

Theorem 2.3 Let Gϕ be a simple, mixed, and a finite graph, while the underlying graph G is a random graph Gn(p) at the same time, and the cardinal numbers of sets of undirected edges and directed edges satisfy equation (5). Thus, NE(Gϕ), the network energy of Gϕ enjoys the inequality as follows[6]:

$$NE(G^{\phi}) < \frac{\sqrt{2}}{2} n^{3/2} \sqrt{p(2-r)} \tag{12}$$

Theorem 2.4 Let G be a simple, undirected, and a finite graph, while a random graph Gn(p) at the same time. Thus, E(Gn(p)), the energy of G enjoys the equation asymptotically almost surely as follows [8]:

$$E(G) = 8n^{3/2}\left[\frac{1}{3\pi}\sqrt{p(1-p)} + o(1)\right] \tag{13}$$

Theorem 2.5 Let Gσ be a simple, oriented, and a finite graph, while the underlying graph G is a random graph Gn(p) at the same time. Thus, εS(Gσ), the skew energy of Gσ enjoys the equation asymptotically almost surely as follows [9]:

$$\varepsilon_S(G^{\sigma}) = 8n^{3/2}\left[\frac{1}{3\pi}\sqrt{p} + o(1)\right] \tag{14}$$

Theorem2.6 Let Gϕ be a simple, mixed, and a finite graph with the maximum degreeΔ. Thus, εH(Gϕ), the Hermitian energy of Gϕ, enjoys the inequality as follows [6]:

$$\begin{cases} \varepsilon_H(G^{\phi}) \le \sqrt{2mn} \le n\sqrt{\Delta} \\ 2\sqrt{m} \le \varepsilon_H(G^{\phi}) \le 2m \end{cases} \tag{15}$$

3. Relations among Network Energies

For undirected graph G, oriented graph Gσ, and mixed graph Gϕ, combing with equations (1), (2) and (6), the Theorem 3.1 can be obtained as follows.

Theorem 3.1 If oriented graph Gσand mixed graph Gϕ share the same underlying graph G, NE(Gσ), NE(Gϕ), and NE(G), the network energy of them enjoy the inequality as follows:

$$1 \le NE(G^{\sigma}) \le NE(G^{\phi}) \le NE(G) \tag{16}$$

with equality iff

$$NE(G^{\sigma}) \cong NE(G^{\phi}) \cong NE(G) \cong \overline{K_n} \tag{17}$$

while

$$\begin{cases} V(G^{\sigma}) = V(G^{\phi}) = V(G) = \varnothing \\ E(G^{\sigma}) = E(G^{\phi}) = E(G) = \varnothing \end{cases} \tag{18}$$

thus

$$\begin{cases} |V(G^{\sigma})| = |V(G^{\phi})| = |V(G)| = 0 \\ |E(G^{\sigma})| = |E(G^{\phi})| = |E(G)| = 0 \end{cases} \tag{19}$$

Combing with equations (10) and (11), the Theorem 3.2 can be obtained as follows.

Theorem3.2 If the underlying graph of oriented graph Gσ is G, at the same time, if n→∞, thus, NE(G) and NE(Gσ), the network energy of G and Gσ enjoy the following equation:

$$\lim_{n\to\infty} \frac{NE(G)}{NE(G^{\sigma})} = \sqrt{2} \tag{20}$$

The Theorem 3.3 can be obtained as follows combing with equations (10) and (12).

Theorem 3.3 If the underlying graph of mixed graph Gϕ is G, at the same time, if n→∞, thus, NE(G) and NE(Gϕ), the network energy of G and Gϕ enjoy the following equation:

ISSN:2790-1688 DOI: 10.56028/aetr.1.1.94

$$\lim_{n\to\infty}\frac{NE(G)}{NE(G^{\phi})}=\frac{\sqrt{2}}{\sqrt{2-r}} \tag{21}$$

The Theorem 3.4 can be obtained as follows combing with equations (11) and (12).

Theorem 3.4 If the oriented graph Gσ and the mixed graph Gϕ share the same underlying graph G, at the same time, if n→∞, the NE(Gσ) and NE(Gϕ) enjoy the following equation:

$$\lim_{n\to\infty}\frac{NE(G^{\sigma})}{NE(G^{\phi})}=\frac{1}{\sqrt{2-r}} \tag{22}$$

4. Relations among Energy, Skew Energy, and Hermitian Energy

The Theorem 4.1 can be obtained as follows by Theorem 2.4 with equation (13).

Theorem 4.1 If the graph G is a random graph Gn(p), thus, E(Gn(p)), the energy of G, enjoys the inequality as follows:

$$E(G)\le\frac{4}{3\pi}n^{3/2} \tag{23}$$

with equality iff $p=\frac{1}{2}$.

Proof: For

$$\begin{aligned}E(G)&=8n^{3/2}\left[\frac{1}{3\pi}\sqrt{p(1-p)}+o(1)\right]\\&=8n^{3/2}\left[\frac{1}{3\pi}\sqrt{p-p^{2}}+o(1)\right]\\&=8n^{3/2}\left[\frac{1}{3\pi}\sqrt{\frac{1}{4}-(p^{2}-p+\frac{1}{4})}+o(1)\right]\\&=8n^{3/2}\left[\frac{1}{3\pi}\sqrt{\frac{1}{4}-(p-\frac{1}{2})^{2}}+o(1)\right]\\&\le 8n^{3/2}\left(\frac{1}{3\pi}\times\frac{1}{2}\right)\\&=\frac{4}{3\pi}n^{3/2}\end{aligned} \tag{24}$$

which completes this proof.

Theorem 4.2 If G1, the graph is a random graph Gn(p), and at the same time, G2, the graph is also a random graph Gn(q), if p+q=1, thus, E(Gn(p)) and E(Gn(q)), the energies of random graphs Gn(p) and Gn(q) enjoy the following equation:

$$E(G_n(p))=E(G_n(q)) \tag{25}$$

Proof: For

$$p=\frac{1}{2}[(p+q)+(p-q)]=\frac{1}{2}+\frac{1}{2}(p-q) \tag{26}$$

$$q=\frac{1}{2}[(p+q)-(p-q)]=\frac{1}{2}-\frac{1}{2}(p-q) \tag{27}$$

Combing with equations (13) and (26), we have

ISSN:2790-1688 DOI: 10.56028/aetr.1.1.94

$$
\begin{aligned}
E(G_n(p)) &= 8n^{3/2}\left[\frac{1}{3\pi}\sqrt{p(1-p)}+o(1)\right] \\
&= 8n^{3/2}\left\{\frac{1}{3\pi}\sqrt{\left[1-(\frac{1}{2}+\frac{1}{2}(p-q)\right]\left[\frac{1}{2}+\frac{1}{2}(p-q)\right]p}+o(1)\right\} \\
&= 8n^{3/2}\left\{\frac{1}{3\pi}\sqrt{\left[\frac{1}{2}-\frac{1}{2}(p-q)\right]\left[\frac{1}{2}+\frac{1}{2}(p-q)\right]p}+o(1)\right\} \\
&= 4n^{3/2}\left[\frac{1}{3\pi}\sqrt{1-(p-q)^2}+o(1)\right]
\end{aligned} \tag{28}
$$

Combing with equations (13) and (27), we have

$$
\begin{aligned}
E(G_n(q)) &= 8n^{3/2}\left[\frac{1}{3\pi}\sqrt{p(1-p)}+o(1)\right] \\
&= 8n^{3/2}\left\{\frac{1}{3\pi}\sqrt{\left[1-(\frac{1}{2}-\frac{1}{2}(p-q)\right]\left[\frac{1}{2}-\frac{1}{2}(p-q)\right]p}+o(1)\right\} \\
&= 8n^{3/2}\left\{\frac{1}{3\pi}\sqrt{\left[\frac{1}{2}+\frac{1}{2}(p-q)\right]\left[\frac{1}{2}-\frac{1}{2}(p-q)\right]p}+o(1)\right\} \\
&= 4n^{3/2}\left[\frac{1}{3\pi}\sqrt{1-(p-q)^2}+o(1)\right]
\end{aligned} \tag{29}
$$

Combing with equations (28) and (29), we can obtain equation (25).
Thus complete this proof.

Lemma 4.1 The energy E(G) and $E(\overline{G})$ enjoy the following equation:

$$E(G)=E(\overline{G}) \tag{30}$$

Proof: For

$$G+\overline{G}=K_n \tag{31}$$

so G can be considered as the edges of it are connected of p with

$$p=\frac{2\left|E(G)\right|}{\left|V(G)\right|\left(\left|V(G)\right|-1\right)} \tag{32}$$

and $\overline{G}$ can be considered as the edges of it are connected of q with

$$q=\frac{2\left|E(\overline{G})\right|}{\left|V(\overline{G})\right|\left(\left|V(\overline{G})\right|-1\right)} \tag{33}$$

while $V(G)=V(\overline{G})$, thus

$$E(G)+E(\overline{G})=\frac{1}{2}\left|V(G)\right|\left(\left|V(G)\right|-1\right)=\frac{1}{2}\left|V(\overline{G})\right|\left(\left|V(\overline{G})\right|-1\right) \tag{34}$$

In this way, p+q=1, combing with equations (25), we can obtain equation (30) immediately.
The proof is thus completed.

The Theorem 4.3 can be obtained as follows combing with equations (13) and (14).

Theorem 4.3 If the underlying graph of oriented graph Gσ is G, and, at the same time, if n→∞, the E(G) and εS(Gσ) enjoy the following equation:

$$\lim_{n\to\infty}\frac{E(G)}{\varepsilon_S(G^\sigma)}=\sqrt{1-p} \tag{35}$$

Combing with equations (13), (14), and (25), we have Theorem 4.4 as follows.

ISSN:2790-1688 DOI: 10.56028/aetr.1.1.94

Theorem 4.4 If Gσ1 is a simple, oriented, and a finite graph, and G1, the underlying graph of Gσ1 is a random graph Gn(p), while Gσ2 is also a simple, oriented, and a finite graph, and G2, the underlying graph of Gσ2 is a random graph Gn(q), if p+q=1, at the same time, thus, εS(Gσ1) and εS(Gσ2), the skew energies of Gσ1 and Gσ2, and E(Gn(p)) and E(Gn(q)), the energies of Gn(p) and Gn(q), enjoy the following equation:

$$\begin{cases} \varepsilon_S(G^\sigma{}_1)\varepsilon_S(G^\sigma{}_2) = 8n^{3/2}E(G_n(p)) \\ \varepsilon_S(G^\sigma{}_1)\varepsilon_S(G^\sigma{}_2) = 8n^{3/2}E(G_n(q)) \end{cases} \tag{36}$$

Proof: For

$$\begin{cases} \varepsilon_S(G^\sigma{}_1) = 8n^{3/2}\left[\dfrac{1}{3\pi}\sqrt{p} + o(1)\right] \\ \varepsilon_S(G^\sigma{}_2) = 8n^{3/2}\left[\dfrac{1}{3\pi}\sqrt{q} + o(1)\right] \end{cases} \tag{37}$$

Combing with equations (13) and (37), thus

$$\begin{aligned} &\varepsilon_S(G^\sigma{}_1)\varepsilon_S(G^\sigma{}_2) \\ &= 8n^{3/2}\left[\frac{1}{3\pi}\sqrt{p} + o(1)\right] \times 8n^{3/2}\left[\frac{1}{3\pi}\sqrt{q} + o(1)\right] \\ &= 8n^{3/2} \times 8n^{3/2}\left[\frac{1}{3\pi}\sqrt{p} + o(1)\right]\left[\frac{1}{3\pi}\sqrt{q} + o(1)\right] \\ &= 8n^{3/2} \times 8n^{3/2}\left[\frac{1}{3\pi}\sqrt{pq} + o(1)\right] \\ &= 8n^{3/2} \times 8n^{3/2}\left[\frac{1}{3\pi}\sqrt{p(1-p)} + o(1)\right] \\ &= 8n^{3/2}E(G_n(p)) = 8n^{3/2}E(G_n(q)) \end{aligned} \tag{38}$$

The proof is thus complete.

5. Relations among Network Energies and Others Energies

For all energies, e.g., network energy, energy, skew energy, Hermitian energy, etc., are estimated on random undirected graphs, random oriented graphs, and random mixed graphs, the relations among previous energies become another focus that appear in front of us. Energy, skew energy, and Hermitian energy will be compared with network energy within that context in this section.

The Theorem 5.1 can be obtained as follows combing with equations (10) and (13).

Theorem 5.1 Let G be a simple, undirected, and a finite graph, and a random graph Gn(p), at the same time, if n→∞, thus, E(Gn(p)) and NE(Gn(p)), the energy and network energy of G enjoy the following equation:

$$\lim_{n\to\infty}\frac{E(G)}{NE(G)} = \frac{8}{3\pi}\sqrt{1-p} \tag{39}$$

The Theorem 5.2 can be obtained as follows combing with equations (11) and (14).

Theorem 5.2 Let Gσ be a simple, oriented, and finite graph, and G, the underlying graph, is a random graph Gn(p), at the same time, if n→∞, εS(Gσ) and NE(Gσ), the skew energy and network energy of Gσ enjoy the following equation:

$$\lim_{n\to\infty}\frac{\varepsilon_S(G^\sigma)}{NE(G^\sigma)} = \frac{8}{3\pi}\sqrt{2} \tag{40}$$

The Theorem 5.3 can be obtained as follows combing with equations (10) and (14).

ISSN:2790-1688 DOI: 10.56028/aetr.1.1.94

Theorem 5.3 Let Gσ be a simple, oriented, and finite graph, and the underlying graph G is a random graph Gn(p), at the same time, if n→∞, εS(Gσ) and NE(G), the skew energy of Gσ and network energy of G enjoy the following equation:

$$\lim_{n\to\infty}\frac{\varepsilon_S(G^\sigma)}{NE(G)}=\frac{8}{3\pi} \tag{41}$$

6. Conclusion

Random graph theory is one of the most important theory of graph theory and complex network from beginning. Energy of undirected graph is the basic theory of graph energy. What's more, energy, skew energy, and Hermitian energy are basic theory of undirected graph, oriented graph, and mixed graph respectively. In this paper, network energy, energy, skew energy, and Hermitian energy of random undirected graph, random oriented graph, and random mixed graph, are computed and compared, and relations among them are obtained. The relations among network energy and other energies of random graphs, and at the same time the relations among network energy and other energies of small world graph, scale free graph, etc, will be our next work.

Acknowledgments

This work is supported by Program for Innovation Research Groups at Institutions of Higher Education in Chongqing (Grant No. CXQT21032), Natural Science Foundation of Chongqing, China (Grant No. cstc2021jcyj-msxmX0532), and the Science and Technology Research Program of Chongqing Municipal Education Commission (Grant Nos. KJQN202103404, KJQN202103401).

References

[1] Liu Shengjiu, Li Tianrui, Zhu Jie, Wang Guoqiang, Liu Jia, and Xie Peng, "Network energy: a new energy of a graph," Proc. IEEE 14th International Conference on Intelligent Systems and Knowledge Engineering(ISKE 2019), IEEE Press, Nov. 2019, pp. 785-789, doi: 10.1109/ISKE47853.2019.9170462.

[2] Liu Shengjiu, Li Tianrui, Zhang Xiaobo, Liu Jia, Xie Peng, and Wang Guoqiang, "On network energy of oriented graphs," Proc. 14th International FLINS Conference on Robotics and Artificial Intelligence/IEEE 15th International Conference on Intelligent Systems and Knowledge Engineering (FLINS 2020/ISKE 2020), World Scientific, Aug. 2020, pp. 11-18, doi: 10.1142/9789811223334_0002.

[3] Liu Shengjiu, Li Tianrui, Xie Peng, and Liu Jia, "Research and Application of Network Energy on Mixed Graphs," Journal of Hunan University(Natural Sciences), vol. 48, Jun. 2021, pp. 105-111, doi: 10.16339/j.cnki.hdxbzkb.2021.06.015.

[4] I. Gutman. "The energy of a graph," Ber. Math. Statist. Sekt. Forschungsz. Graz, vol. 103, 1978, pp. 1-22.

[5] C. Adiga, R. Balakrishnan, and W. So, "The skew energy of a digraph," Linear Algebra & Its Applications, vol. 432, Mar. 2010, pp. 1825-1835, doi:10.1016/j.laa.2009.11.034

[6] Liu Jianxi, and Li Xueliang, "Hermitian-adjacency matrices and Hermitian energies of mixed graphs," Linear Algebra & Its Applications, vol. 466, Feb. 2015, pp. 182-207, doi:10.1016/j.laa.2014.10.028.

[7] S. Erdo, and A. Renyi, "On random graphs I," Publicationes Mathematicae, vol. 6, 1959, pp. 290-297.

[8] Du Wenxue, Li Xueliang Li, and Li Yiyang, "The Laplacian energy of random graphs," Journal of Mathematical Analysis & Applications, vol. 368, Aug. 2010, pp. 311-319, doi:10.1016/j.jmaa.2010.02.051.

[9] Chen Xiaolin, Li Xueliang, and Lian Huishu, "The skew energy of random oriented graphs," Linear Algebra & Its Applications, vol. 438, Jun. 2013, pp.4547-4556, doi:10.1016/j.laa.2013.02.00