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Optimal Capital Structure under Stochastic Interest Rates with Endogenous Default Barriers

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Abstract. Based on the principle of maximization of the utility value for shareholders, we establish an optimal capital structure model under stochastic interest rates with improved endogenous default barriers by considering the tax and bankruptcy risk. From the numerical results, we find the drift and volatility of the firm's log return, the average risk aversion of all the shareholders, the long term mean level of interest rate and the bond maturity are the key variables in determining optimal capital structure. We also find that the utility values behave as a concave function with bond principals. We can conclude that there exists an optimal amount of bond issuance to maximize the utility value of shareholders.

Keywords: capital structure; stochastic interest rate; endogenous default barriers; optimization.

1. Introduction

Capital structure refers to a mix of a company's debt and equity. Under the well-known Modigliani & Miller theorem [1], the capital structure will not influence the firm's value in perfect markets in which there are no taxes, no transaction costs and no bankruptcy costs. But these costs do exist in the real world. Many theories have been put forward to address this. According to [2], trade-off theory, pecking order theory and market timing theory are the most important three. [3] divided the trade-off theory into two parts, one is the static trade-off theory and the other one is dynamic trade-off theory. Representatives of the static trade-off theory are provided by [4] and [5]. [6,7] are both concerned with the dynamic trade-off theory. [8,9] are examples of pecking order theory. market timing theory is developed by [10]. [11] analyzed IPO data of 500 firms loacted in coastal areas listed in China's A-share maket, and the rusults showed that the capital structure of these firms indeed affected by the market timing attempts. Some literatures concern on empirical test, for example, [12] uses empirical analysis method on panel data to verify the EVA effect on capital structure. And [13] illustrated an empirical approach for determing optimal capital structure.

In the trade-off theory, [14] were the first to study the capital structure problem by means of the contingent-claims analysis approach. [15] and [16] extended the contingent-claims method. [7,17] show that a dynamic trade-off model with features that are not typically included in previous capital structure models can explain many stylized facts. [18] provided a number of new insights into capital structure decisions by recognizing that firms simultaneously use different types, sources, and priorities of debt. [19] studied a defaultable firm's debt priority structure in a simple structural model where the firm issues senior and junior bonds and is subject to both liquidity and solvency risks. [20] developed a model in which optimal capital structure and debt maturity are jointly determined in a stochastic interest rate environment. They found that the optimal proportion of debt is smaller than empirical observations. And they also found that the long-run mean of the interest rate is a key variable in determining the optimal capital structure and debt maturity. But they assume that the firm pays tax at a constant tax rate, dis- regarding whether the final firm value is higher or lower than the initial firm value. This does not agree with reality, since a firm in the real world would only have to pay tax when its value becomes larger than the initial value. The same problem also exists in [19]. [21] presents a new "capital structure substitution" theory that is based on one simple hypothesis: the company management manipulate the capital structure so that the earnings per share are maximized.

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In this paper, similar to [20, 21], we build an optimal capital structure model based on the principle of maximization of the shareholders' utility value under an endogenous varying default barrier. But in contrast to these papers, the tax is assumed to be paid at the end of the debt term, and only when the final value of the firm is larger than the initial value, as in the real world. Debt financing generally offers lower costs than the income of the firm's value and will benefit the shareholders' profit. However, an increase of debt will amplify the firm's credit risk, which will reduce the utility of the shareholders. The optimal capital structure in this paper is set to be the best proportion of debt over total firm value that maximizes the shareholders' utility. So the owners of the firm, that is, the shareholders, have to judge how much debt they should allow to maximize their utility per share.

This rest of the paper is organized as follows: Section 2 describes the basic assumptions and the model framework. Section 3 gives some numerical results. Section 4 concludes. Section 5 provides the proofs of some formulas stated in the paper.

2. Assumptions and Model Setting

2.1 Basic Assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is the sample space, $\mathcal{F}$ is a σ-field on Ω, and P is the physical probability measure on Ω and $\mathcal{F}$. We do not assume the financial market is complete. Therefore, there exists a set of equivalent martingale measures. In reality, we need to calibrate to market data to find the risk neutral measure $\mathbb{Q}$ under which we should price contingent claims.

We consider a firm which will issue a T year maturity bond, with the coupon paid at maturity, aiming to maximise the shareholders' utility. The utility function is as follows:

$$U = E^{\mathbb{P}}(X) - \frac{1}{2}\overline{A} \times D^{\mathbb{P}}(X) \tag{1}$$

where X is the after-tax present firm value per share given by formula (18), and $\overline{A}$ is the average risk aversion of the shareholders. $E^{\mathbb{P}}(X)$ and $D^{\mathbb{P}}(X)$ represent the expectation and variance of X under the probability measure $\mathbb{P}$.

2.2 Assumptions and Notation of the Model

Assumption 1: The market interest rate follows the Vasicek model [22]:

$$dr_t = \beta(\alpha - r_t)dt + \sigma_r dW_{rt}^{\mathbb{P}} \tag{2}$$

where α is the long term mean level, $\beta$is the speed of reversion, $\sigma_r$is instantaneous volatility, and $W_{rt}^{\mathbb{P}}$ is a standard Brownian motion under $\mathbb{P}$.

Assumption 2: The before-tax asset value of an unleveraged firm follows a geometric Brownian motion (GBM) process under the physical probability measure $\mathbb{P}$:

$$\frac{dV_t}{V_t} = \mu dt + \sigma_v dW_{vt}^{\mathbb{P}}, \tag{3}$$

where μ is the drift term, σ_v is the constant volatility and $W_{vt}^{\mathbb{P}}$ is a standard Wiener process under $\mathbb{P}$ such that the correlation between $W_{vt}^{\mathbb{P}}$ and $W_{rt}^{\mathbb{P}}$ is ρ.

Under the risk-neutral probability measure $\mathbb{Q}$, we have

$$dW_{vt}^{\mathbb{Q}} = dW_{vt}^{\mathbb{P}} - \frac{r_t - \mu}{\sigma_v}dt \tag{4}$$

and the unleveraged firm value will behave as follows:

$$\frac{dV_t}{V_t} = r_t dt + \sigma_v dW_{vt}^{\mathbb{Q}}. \tag{5}$$

For the interest rate process, we can reformulate it as following under measure $\mathbb{P}$:

$$dr_t = \beta(\alpha - r_t)dt + \sigma_r d\left(\rho W_{vt}^{\mathbb{P}} + \sqrt{1-\rho^2}\overline{W}_{rt}^{\mathbb{P}}\right), \tag{6}$$

in which $Cov(W_{vt}^{\mathbb{P}}, \overline{W}_{rt}^{\mathbb{P}}) = 0$.

Under the risk-neutral measure, we can calibrate to real-world risk-free bond prices to find a relationship between $\overline{W}_{rt}^{\mathbb{P}}$ and $\overline{W}_{rt}^{\mathbb{Q}}$, that is:

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$$d\overline{W}_{rt}^{\mathbb{Q}} = \overline{W}_{rt}^{\mathbb{P}} + \lambda \mathrm{dt}. \tag{7}$$

Here we just assume the parameter λ in our numerical calculation rather than calibrate to real-world risk-free bond prices.

As a result, the interest rate process under $\mathbb{Q}$ should be as:

$$\begin{aligned} r_t &= \left(\beta(\alpha - r_t) + \rho\sigma_r \frac{r_t - \mu}{\sigma_v} - \sqrt{1-\rho^2}\sigma_r\lambda\right)dt + \sigma_r d\left(\rho W_{vt}^{\mathbb{Q}} + \sqrt{1-\rho^2}\overline{W}_{rt}^{\mathbb{Q}}\right) \\ &= \beta_{\mathbb{Q}}\left(\alpha_{\mathbb{Q}} - r_t\right)dt + \sigma_r dW_{rt}^{\mathbb{Q}} \end{aligned} \tag{8}$$

with $\beta_{\mathbb{Q}} = \frac{\beta\sigma_v - \rho\sigma_r}{\sigma_v}$ and $\alpha_{\mathbb{Q}} = \frac{\rho\sigma_r\mu - \beta\alpha\sigma_v + \sqrt{1-\rho^2}\lambda\sigma_v\sigma_r}{\rho\sigma_r - \beta\sigma_v}$.

Under the Vasicek model, the discounted value of a risk-free zero-coupon bond with unit face value and maturity T is:

$$\Gamma(r_t, t; T) = exp(A(t,T) - B(t,T)r_t), \tag{9}$$

where $A(t,T) = \left(\alpha_{\mathbb{Q}} - \frac{\sigma_r^2}{2\beta_{\mathbb{Q}}^2}\right)\left(B(t,T) - (T-t)\right) - \frac{\sigma_r^2 B^2(t,T)}{4\beta_{\mathbb{Q}}}$, $B(t,T) = \frac{1}{\beta_{\mathbb{Q}}}\left(1 - e^{-\beta_{\mathbb{Q}}(T-t)}\right)$.

Assumption 3: If leveraged, the firm will issue a coupon bond with principal P and a finite maturity time T. The coupon is assumed to be paid only once at maturity, and is chosen (see formula (17)) to make the present value of the bond equal to the principal P. We simply assume the bond issuance cost is equal to a fixed cost IC plus a flexible cost which is proportional to the principal P at a constant rate of κ. Let $\mathrm{BIC}(0, \mathrm{P})$ denote the bond issuance cost at time 0, then $\mathrm{BIC}(0, \mathrm{P}) = \mathrm{IC} + \kappa\mathrm{P}$.

The firm will have to pay the bond issuance cost first, so the initial value of the firm will become $\widehat{\mathrm{V}}_0 = \mathrm{V}_0 - (\mathrm{IC} + \kappa\mathrm{P})$. The leveraged firm value process under measure $\mathbb{P}$ and measure $\mathbb{Q}$ should be as follows with initial value $\widehat{\mathrm{V}}_0$:

$$\frac{d\widehat{V}_t}{\widehat{V}_t} = \mu dt + \sigma_v dW_{vt}^{\mathbb{P}} \tag{10}$$

and

$$\frac{d\widehat{V}_t}{\widehat{V}_t} = r_t dt + \sigma_v dW_{vt}^{\mathbb{Q}}. \tag{11}$$

Assumption 4: We assume that the firm pays tax only at time T and the tax rate is θ. The tax will only be paid when the final firm value is larger than the initial value. We denote the leveraged after-tax firm value at time T as $\widehat{\mathrm{V}}_{\mathrm{T}}^*$, then

$$\widehat{V}_T^* - \theta\left(\widehat{V}_T - V_0\right)^+. \tag{12}$$

For unleveraged firm, $\widehat{V}_T$ should equal to V_T.

Assumption 5: Default Barriers

We assume that bankruptcy occurs when the firm's asset value hits an endogenous default barrier $\left\{\widehat{\mathrm{V}}_{\mathrm{t}}^{\mathrm{B}}, \mathrm{t} = 0\!:\!\mathrm{T}\right\}$. We define the default barrier at which the expected discounted after-tax firm value at maturity under the risk-neutral probability measure $\mathbb{Q}$ will be below the discount value of the debt principal, that is:

$$E^{\mathbb{Q}}\left(\widehat{V}_T^* e^{-\int_t^T r_s ds} | \mathcal{F}_t\right) < P \times \Gamma(r_t, t; T). \tag{13}$$

This inequality can equivalently be written as $\widehat{\mathrm{V}}_{\mathrm{t}} < \widehat{\mathrm{V}}_{\mathrm{t}}^{\mathrm{B}}(\mathrm{r}_{\mathrm{t}}, \mathrm{t}; \mathrm{P}, \mathrm{T})$ with $\widehat{\mathrm{V}}_{\mathrm{t}}^{\mathrm{B}}(\mathrm{r}_{\mathrm{t}}, \mathrm{t}; \mathrm{P}, \mathrm{T})$ given by (14).

Following formula (19) in [23], we have

$$\begin{aligned} E^{\mathbb{Q}}\left(\widehat{V}_T^* e^{-\int_t^T r_s ds} | \mathcal{F}_t\right) &= E^{\mathbb{Q}}\left(\left[\widehat{V}_T - \theta\left(\widehat{V}_T - V_0\right)^+\right] e^{-\int_t^T r_s ds} | \mathcal{F}_t\right) \\ &= E^{\mathbb{Q}}\left(\widehat{V}_t e^{\int_t^T r_s ds - \frac{1}{2}\sigma_v^2(T-t) + \sigma_v W_{v(T-t)}^{\mathbb{Q}}} e^{-\int_t^T r_s ds} | \mathcal{F}_t\right) \\ &\quad -\theta E^{\mathbb{Q}}\left(\left(\widehat{V}_t e^{\int_t^T r_s ds - \frac{1}{2}\sigma_v^2(T-t) + \sigma_v W_{v(T-t)}^{\mathbb{Q}}} - V_0\right)^+ e^{-\int_t^T r_s ds} | \mathcal{F}_t\right) \end{aligned}$$

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$$= E^{\mathbb{Q}}\left(\hat{V}_t e^{-\frac{1}{2}\sigma_v^2(T-t)+\sigma_v W^{\mathbb{Q}}_{v(T-t)}}|\mathcal{F}_t\right) - \theta E^{\mathbb{Q}}\left(\left(\hat{V}_t e^{\int_t^T r_s ds - \frac{1}{2}\sigma_v^2(T-t)+\sigma_v W^{\mathbb{Q}}_{v(T-t)}} - V_0\right)^+ e^{-\int_t^T r_s ds}|\mathcal{F}_t\right)$$
$$= \hat{V}_t - \theta C(\hat{V}_t, T-t, V_0)$$

where

$$\begin{cases} C(\hat{V}_t, T-t, V_0) = \hat{V}_t N(d_1) - V_0 \Gamma(r_t, t; T) N(d_2) \\ d_1(t, T, r_t, \hat{V}_t) = \frac{\log\left(\frac{\hat{V}_t}{V_0 \Gamma(r_t,t;T)}\right) + \frac{1}{2}\Sigma^2(T-t)}{\sqrt{\Sigma^2(T-t)}} \\ d_2(t, T, r_t, \hat{V}_t) = d_1(t, T, r_t, \hat{V}_t) - \sqrt{\Sigma^2(T-t)} \\ \Sigma^2(T-t) = (T-t)\sigma_v^2 - \frac{\sigma_r^2}{\beta_{\mathbb{Q}}^3}\left(-\beta_{\mathbb{Q}}(T-t) + \frac{3}{2} - 2e^{-\beta_{\mathbb{Q}}(T-t)} + \frac{1}{2}e^{-2\beta_{\mathbb{Q}}(T-t)}\right) \\ \qquad + \frac{2\rho\sigma_v\sigma_r}{\beta_{\mathbb{Q}}^2}\left(e^{-\beta_{\mathbb{Q}}(T-t)} + \beta_{\mathbb{Q}}(T-t) - 1\right). \end{cases}$$

Then $E^{\mathbb{Q}}\left(\hat{V}_T^* e^{-\int_t^T r_s ds}|\mathcal{F}_t\right) < P \times \Gamma(r_t, t; T)$ implies that

$\hat{V}_t - \theta \hat{V}_t N\left(d_1(t, T, r_t, \hat{V}_t)\right) + \theta V_0 \Gamma(r_t, t; T) N\left(d_2(t, T, r_t, \hat{V}_t)\right) < P \times \Gamma(r_t, t; T)$,which gives $\hat{V}_t - \theta \hat{V}_t N\left(d_1(t, T, r_t, \hat{V}_t)\right) + \theta V_0 \Gamma(r_t, t; T) N\left(d_2(t, T, r_t, \hat{V}_t)\right) - P \times \Gamma(r_t, t; T) < 0.$

Denote

$\Pi(t, T, \hat{V}_t, P) = \hat{V}_t - \theta \hat{V}_t N\left(d_1(t, T, r_t, \hat{V}_t)\right) + \theta V_0 \Gamma(r_t, t; T) N\left(d_2(t, T, r_t, \hat{V}_t)\right) - P \times \Gamma(r_t, t; T).$

We find that $\frac{\partial \Pi(t, T, \hat{V}_t, P)}{\partial \hat{V}_t} > 0$, it means that $\Pi(t, T, \hat{V}_t, P)$ is a monotonous function on $\hat{V}_t$ and have a unique solution on $\Pi(t, T, \hat{V}_t, P) = 0$. So we define:

$$\hat{V}_t^B(r_t, t; P, T) = \{\hat{V}_t: \Pi(t, T, \hat{V}_t, P) = 0\}. \tag{14}$$

Definition: First passage time

We define

$$\tau = min[t: \hat{V}_t \leq \hat{V}_t^B(r_t, t; P, T)], \tag{15}$$

which is the first time at which the asset value $\hat{V}_t$ hits $\hat{V}_t^B(r_t, t; P, T)$.

The firm should not default at time 0, which means τ should be larger than 0. That is $\hat{V}_0 > \hat{V}_0^B$, which implies $V_0 - (IC + \kappa P) - \frac{\left[P - \theta V_0 N\left(d_2(0, T, r_0, \hat{V}_0)\right)\right]\Gamma(r_0, 0; T)}{1 - \theta N\left(d_1(0, T, r_0, \hat{V}_0)\right)} > 0.$

So we have $P < \frac{(V_0 - IC)\left(1 - \theta N\left(d_1(0, T, r_0, \hat{V}_0)\right)\right) + \theta V_0 N\left(d_2(0, T, r_0, \hat{V}_0)\right)\Gamma(r_0, 0; T)}{\Gamma(r_0, 0; T) + \kappa\left(1 - \theta N\left(d_1(0, T, r_0, \hat{V}_0)\right)\right)}$. Moreover, P should be never be larger than the initial firm value, so $P < \hat{V}_0 = V_0 - (IC + \kappa P)$, which means $P < \frac{V_0 - IC}{1 + \kappa}$.

As a result, $P < min\left(\frac{V_0 - IC}{1 + \kappa}, \frac{(V_0 - IC)\left(1 - \theta N\left(d_1(0, T, r_0, \hat{V}_0)\right)\right) + \theta V_0 N\left(d_2(0, T, r_0, \hat{V}_0)\right)\Gamma(r_0, 0; T)}{\Gamma(r_0, 0; T) + \kappa\left(1 - \theta N\left(d_1(0, T, r_0, \hat{V}_0)\right)\right)}\right)$ or else the firm will default immediately at time 0.

2.3 Maximization of Shareholders' Utility Value

2.3.1. Calculate the coupon CP at maturity

Before calculating the coupon, we should first find out what will influence the bond value. The bond cash flows at any time s are as follows:

$$\mathrm{d}(s) = \begin{cases} P + C_P & s = T \text{ and } T < \tau \\ \left[(1 - \emptyset)\hat{V}_s^B - \theta\left((1 - \emptyset)\hat{V}_s^B - V_0\right)^+\right] & s = \tau \leq T. \end{cases}$$

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The second line means that when the firm goes bankrupt, it should pay the clearing fee (proportional to the firm value at default at a recovery rate of $\emptyset$) first and then pay tax. So the present bond value should be equal to the expectation of the sum of all the discount cash flows under the risk-neutral probability measure $\mathbb{Q}$ (we denote the present value of debt by $\mathrm{DV}(\mathrm{T}, \mathrm{V}_0, \mathrm{r}_0, \mathrm{P})$):

$$DV(T, V_0, r_0, P) = (P + C_P)E_0^{\mathbb{Q}}\left(1_{\tau>T}e^{-\int_0^T r_u du}\right) + E_0^{\mathbb{Q}}\left(\int_0^T \left[(1-\emptyset)\widehat{V}_s^B - \theta\left((1-\emptyset)\widehat{V}_s^B - V_0\right)^+\right] \times \delta(s-\tau)\, 1_{\tau<T}e^{-\int_0^s r_u du}ds\right) = (P + C_P)E_0^{\mathbb{Q}}\left(1_{\tau>T}e^{-\int_0^T r_u du}\right) + E_0^{\mathbb{Q}}\left(\int_0^T \left[(1-\emptyset)\widehat{V}_s^B - \theta\left((1-\emptyset)\widehat{V}_s^B - V_0\right)^+\right] \times \delta(s-\tau)\, e^{-\int_0^s r_u du}ds\right),$$

where $\delta(s-\tau) = \begin{cases} 1 & s = \tau \\ 0 & s \neq \tau \end{cases}$.

Let

$$\begin{cases} L(T, V_0) = E_0^{\mathbb{Q}}\left(1_{\tau>T}e^{-\int_0^T r_u du}\right) \\ H(T, V_0) = E_0^{\mathbb{Q}}\left(\int_0^T \left[(1-\emptyset)\widehat{V}_s^B - \theta\left((1-\emptyset)\widehat{V}_s^B - V_0\right)^+\right] \times \delta(s-\tau)\, e^{-\int_0^s r_u du}ds\right). \end{cases}$$

So we get:

$$P = DV(T, V_0, r_0, P) = (P + C_P) \times L(T, V_0) + H(T, V_0). \tag{16}$$

which implies

$$C_P = \frac{P - H(T, \widehat{V}_0) - P \times L(T, V_0)}{L(T, V_0)}. \tag{17}$$

We have to compute $\mathrm{L}(\mathrm{T}, \mathrm{V}_0)$ and $\mathrm{H}(\mathrm{T}, \mathrm{V}_0)$ numerically in Section 3 because of the complexity of τ.

2.3.2 Shareholders' utility function

We now consider the after-tax leveraged firm value at time T from the perspective of the stock shareholders.

If the firm has not defaulted till time T, the final value belonging to the shareholders should be equal to the final leveraged firm value minus the coupon payment value at time T (that is $\mathrm{C_P}$, which has been given as formula (17)), the tax and the principal of the bond (P). If the firm defaults before time T, then at the default time the firm should pay a clearing fee first and then pay tax. Any value left after that should be all paid to the bondholders. As a result, nothing will be left to the shareholders. So to the shareholders, once upon default, they do not care about how much the recovery be since they will receive nothing in any case.

We denote the after-tax leveraged firm value belonging to the shareholders at time T by $\widehat{\mathrm{V}}_{\mathrm{T}}^{\mathrm{S}}$:

$$\widehat{V}_T^S = \begin{cases} \left(\widehat{V}_T - C_P - \theta\left(\widehat{V}_T - C_P - V_0\right)^+ - P\right)^+ & if\ \tau > T \\ 0 & if\ \tau \leq T. \end{cases}$$

Here we take the positive part in the first line because according to the barriers' definition, $\widehat{\mathrm{V}}_{\mathrm{T}}^{\mathrm{B}}$ should equal to P, which means when $\widehat{\mathrm{V}}_{\mathrm{T}} > \mathrm{P}$, and the firm never defaults before time T, τ should be larger than T. But on the other hand, $\widehat{\mathrm{V}}_{\mathrm{T}} - \mathrm{C_P} - \theta\left(\widehat{\mathrm{V}}_{\mathrm{T}} - \mathrm{C_P} - \mathrm{V}_0\right)^+ - \mathrm{P}$ have the chance to less than 0 even when $\widehat{\mathrm{V}}_{\mathrm{T}} > \mathrm{P}$. At this situation, the stockholders will get nothing but not receive the negative value.

At time 0, the stock-holders need to judge the level of debt they need for optimum leverage. It means that they will choose the principal amount to maximize the utility value. Denote the after-tax present firm value per share as X:

$$X = \frac{\widehat{V}_T^S e^{-\int_0^T r_s ds}}{V_0 - P}. \tag{18}$$

Then the utility function of the shareholders corresponding to the principal P is:

$$U(P) = E^{\mathbb{P}}\left(\frac{\widehat{V}_T^S e^{-\int_0^T r_s ds}}{V_0 - P}\right) - \frac{1}{2}\bar{A} \times D^{\mathbb{P}}\left(\frac{\widehat{V}_T^S e^{-\int_0^T r_s ds}}{V_0 - P}\right). \tag{19}$$

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That is, they will compute $\max_P U(P)$ to find the optimal bond principal and the corresponding utility. Because of the complexity of the first passage time τ, we cannot get $U(P)$ in closed form, so we have to calculate the values numerically in Section 3.

3. Numerical Results

In this section, we analyse the numerical results based on the parameters in Table 1. For the Vasicek model parameters, we just use the estimated results as [24] . According to formula (16), the maximum amount of bond principal this company can issue is 99.9 (last cell in Table 1).

Table 1. Parameters for simulation

Interest rate	$\beta = 0.099$	$\alpha = 0.101$	$\sigma_r = 0.01$	$r_0 = 0.05$	$\lambda = 0.3$
Firm asset	$\mu = 0.12$	$\sigma_v = 0.15$	$V_0 = 100$	$\rho = 0.1$	$\bar{A} = 1$
Others	$T = 5$ years	$\theta = 0.2$	$\phi = 0.4$		
	$IC = 0.001$	$\kappa = 0.001$	$P^{max} = 99.9$		

We use MATLAB to do the numerical calculation. For a given bond principal P (such as P = 20), we find the corresponding utility by 2 steps:

Step 1: Coupon calculation under risk-neutral measure $\mathbb{Q}$

1) Simulate a $M \times N$ matrix of the leveraged firm values as formula (11) by Monte-Carlo simulation methods, where M represents the paths number, and N represents the time steps;

2) Simulate a $M \times N$ matrix of the interest rates as formula (8);

3) For each path, calculate the unit zero coupon bond values as formula (9) corresponding to every time t and interest rate r_t;

4) For each path, calculate the barriers as formula (14) corresponding to every time t and interest rate r_t;

5) For each path, calculate the first passage time as (15), corresponding to simulated leveraged firm values $\widehat{V}_t$ and calculated default barriers $\widehat{V}_t^B$;

6) Calculate the coupon as formula (17).

Step 2: Utility calculation under real world measure $\mathbb{P}$

1) Simulate a $M \times N$ matrix of the leveraged firm values as formula (10);

2) Simulate a $M \times N$ matrix of the interest rates as formula (2);

3) For each path, calculate the unit zero coupon bond values as formula (9) corresponding to every time t and interest rate r_t;

4) For each path, calculate the barriers as formula (14) corresponding to every time t and interest rate r_t;

5) For each path, calculate the first passage time as (15) corresponding to simulated leveraged firm values $\widehat{V}_t$ and calculated default barriers $\widehat{V}_t^B$;

6) Calculate the utility as formula (19).

We take a grid dividing the interval from 0 to 99.9 into 30 equal parts, and then calculate the utility values at each grid point using formula (19). We draw the curve in Fig. 1 when P is less than 79.92. We can see that the values line increases step by step from $P = 0$, approaches the maximum amount, and then turns downwards. Along with the value of P grow continuously, the utility value move downwards quickly. As a result, we can conclude that there exists an optimal amount of bond issuance when we consider the utility maximization of the shareholders.

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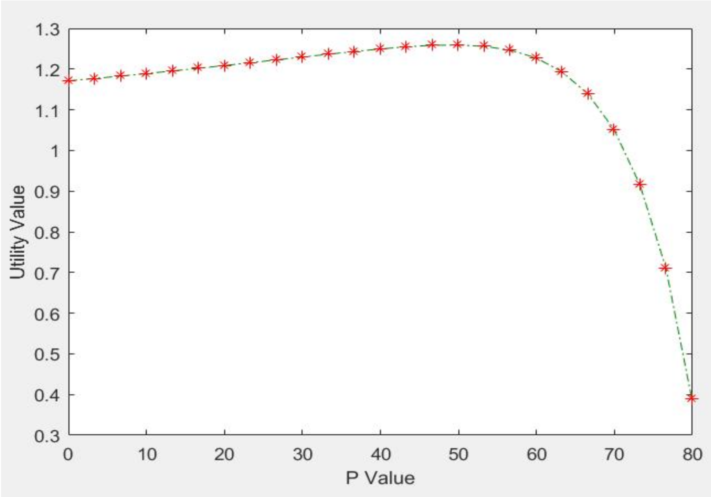

Fig. 1: Utility values corresponding to different principals P ($P^{max} = 79.92$).

In order to identify the relationship between some parameters and the optimal principal, we present some sensitivity tests. For given parameters, we find the optimal principal as follows:

1) Set the interval of principal as [0,99.9];

2) Take a grid dividing the interval into 4 parts, and then calculate the utility values at each grid point as mentioned above;

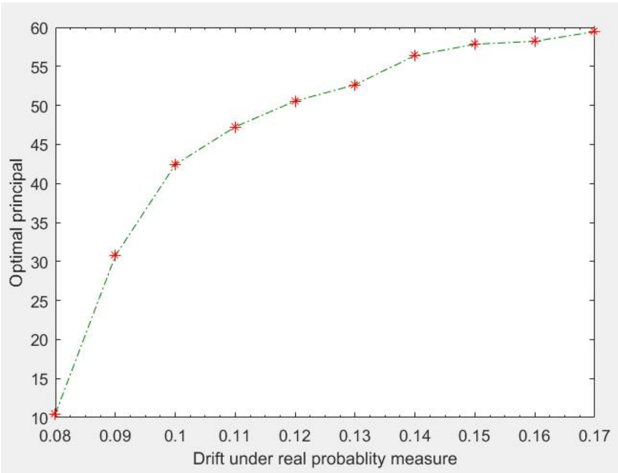

Fig. 2: Optimal bond principal depending on the drift (μ).

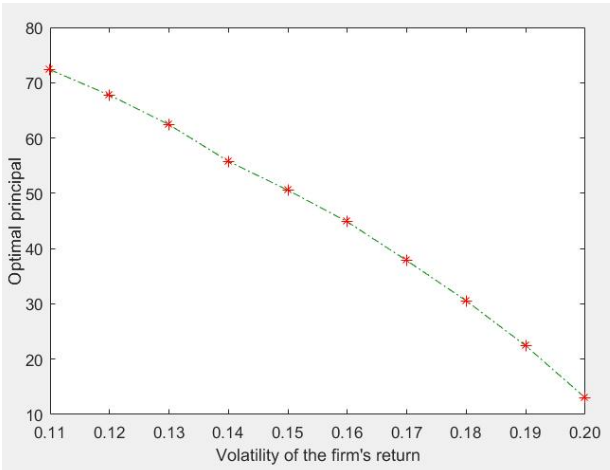

Fig. 3: Optimal bond principal depending on the volatility (σ_v) of the firm's return.

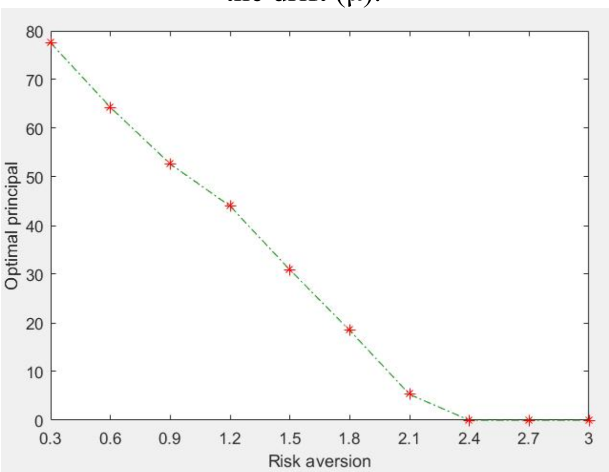

Fig. 4: Optimal bond principal depending on the risk aversion ($\bar{A}$).

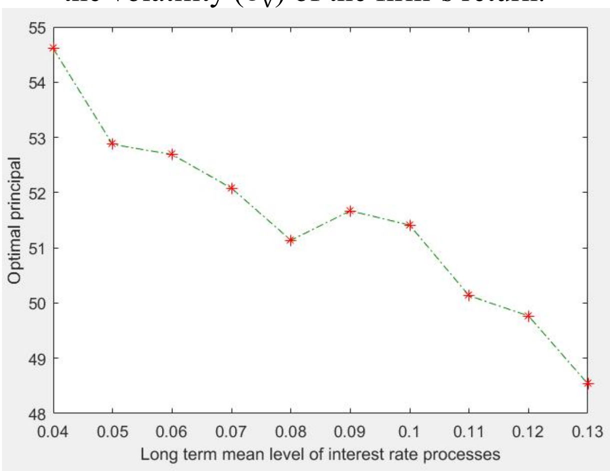

Fig. 5: Optimal bond principal depending on long term mean level (α) of the interest rate process.

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3) Find the grid point corresponding to the largest utility. Set the lower bound of the interval of principal as the front one of this point and the upper bound as the behind one. If this point is the lower bound or the upper bound, then just set the lower bound or the upper bound as itself;

4) Repeat 2) and 3) till the interval is small enough;

5) Take the final grid point corresponding to the largest utility as theoptimal principal.

Fig. 2 shows how the optimal principal behaves as a function of the drift (μ) of the Brownian motion under the real probability measure. When the drift goes up, the optimal principal also goes up but the marginal growth slows down. Of course, the firm with larger drift but the same volatility will make more profit in expectation. And as long as the interest payment cost of the bond is lower than the profit gained, the more of the bond be issued the more profit will be made. When the marginal utility growth caused by the difference between the profit from the firm and the interest payment, and the marginal utility decrease caused by the risk raising by the bond, are equal to each other, the principal will approach to the optimal amount. And since the utility increases with the drift, but decline with the bond principal, the optimal bond principal behaves as a logarithmic function of the drift.

Fig. 3 shows that when the volatility of the firm's returns increases, the corresponding optimal bond principal decreases rapidly. This is so because the greater the volatility, the easier it is for the firm to default, which in turn reduces the utility of the shareholders.

Fig. 4 shows the relationship between average risk aversion and the optimal principal. We have found that when the risk aversion increases from 0.3 to 3 (while all the other parameters are kept constant as in Table 1), the corresponding optimal principal drops rapidly. In particular, when the average risk aversion grows to 2.4 and over, the optimal principal drops to zero, which means that in such a situation the firm will not be issuing any bonds. It is reasonable that the larger the average risk aversion of the shareholders, the lower the attractiveness of the bond since the bond would reduce the utility of the shareholders. And when the average risk aversion becomes large enough, even a small amount of the bond will make the utility less than that in an unleveraged situation.

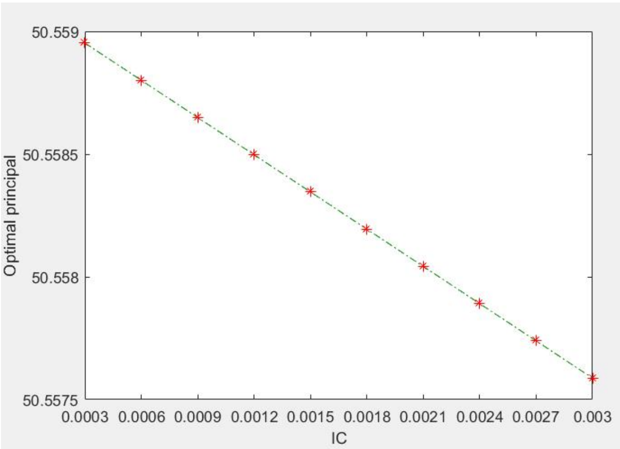

Fig. 6: Optimal bond principal depending on fixed bond issuance cost (IC).

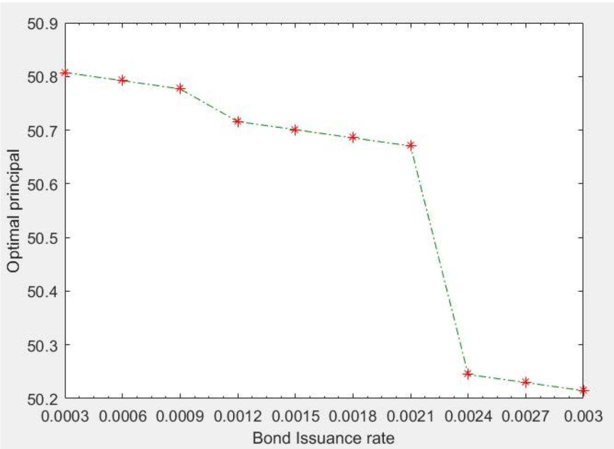

Fig. 7: Optimal bond principal depending on bond proportional issuance cost rate (κ).

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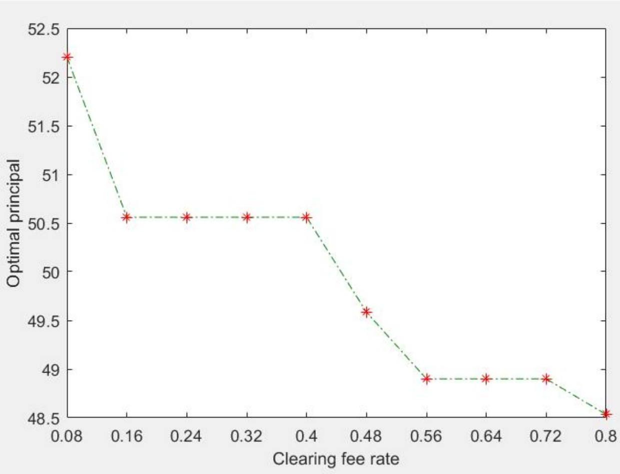

Fig. 8: Optimal bond principal depending on clearing fee rate (ϕ).

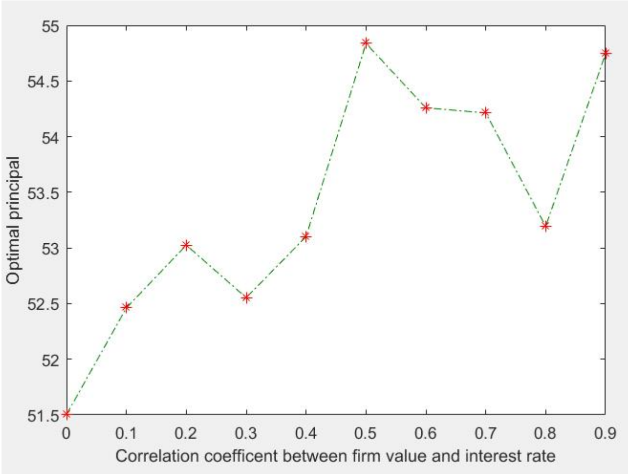

Fig. 9: Optimal bond principal depending on correlation coefficient between firm value and interest rate (ρ).

Fig. 5 shows the relationship between the optimal principal and long term mean level (α) of the interest rate process. When we keep all the other parameters unchanged, the optimal principal decreases as α increases. It is because the interest payment increases along with an increase of market interest rate. The increase of interest payment will reduce the attractiveness of the bond.

Fig. 6 shows the optimal bond principal depending on the fixed bond issuance costs (IC). When the fixed bond issuance costs grow, the corresponding optimal bond principal drops linearly. Since the fixed costs are assumed very small, the optimal principal changes by a small amount too.

Fig. 7 shows the relationship between the issuance cost rate (κ) and the optimal bond principal. An increase of κ means an increase in the issuance cost, which in turn means a decrease of the optimal principals.

Fig. 8 illustrates the relationship between the optimal principal and the clearing fee rate (ϕ). The optimal principal drops with the increase of ϕ. This is consistent with reality when ϕ increases, it means a reduction of the recovery rate for bondholder, who will need more interest to compensate, which in turn, will increase the issuance cost, and finally, reduce the utility.

Fig. 9 shows a weak relationship between the optimal principals and the correlation of firm value and interest rate.

In order to find how the bond maturity influence the capital structure, we conduct the simulation on different bond maturities as Fig. 10. From this figure we find that the optimal principal drops linearly with the increase of the maturity. It is obviously since with the increase of the maturity, the default risk of the firm grows, then the firm have to issue less bond to survive.

4. Conclusions

Based on the principle of maximizing shareholders' utility, considering the tax and bankruptcy risk, this paper constructs an optimal capital structure model with improved endogenous default barriers under stochastic interest rates. The improvement makes the default barrier a function of time t, interest rate rt, firm value $\widehat{V}_t$ and bond principal P.

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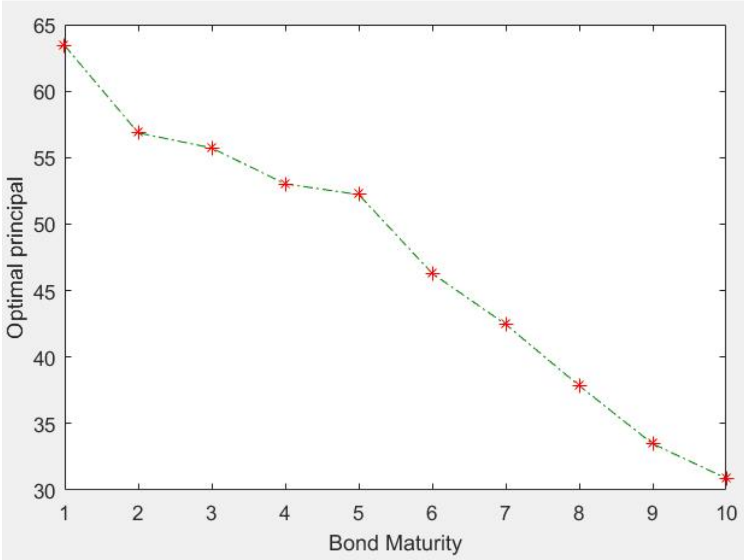

Fig. 10: Optimal bond principal depending on bond maturity (T).

The numerical results show that, along with the value of P growing from 0 to maximum, the utility first increases and then drops. As a result, we can find the optimal bond principal corresponding the maximum utility of shareholders.

Sensitivity tests involving various parameters show that the optimal bond principal increases with the drift (μ) and decreases with volatility (σ_v) of the firm return, the average risk aversion ($\overline{A}$) of the shareholders, the long term mean level (α) of the interest rate process, the fixed bond issuance cost (IC), the bond proportional issuance cost rate (κ), the clearing fee rate (ϕ) and the bond maturity (T). An additional test shows that the optimal principal has no apparent relationship with the correlation coefficient (ρ) between the firm value and interest rate. Among all the parameters, μ, σ_v, $\overline{A}$, α and T are the key variables in determining the optimal capital structure, which is consistent with reality.

For tractability, under the risk-neutral probability, the default barrier is defined so that the expected discounted after-tax firm value at maturity will be below the discount value of the debt principal but not the debt principal plus the coupon payment. This is because the coupon is affected by the default information which is determined by the default barriers. The interaction between the coupon and the default barriers makes it is impossible to get the default barriers' expression implicitly or explicitly.

Our model simply assumes that the firm evolves only over one time period. In [20] the authors developed a model in which the optimal capital structure and debt maturity are jointly determined for multiple time periods in a stochastic interest rate environment. Incorporating these features would be another challenging topic for future research.

Reference

[1] Franco Modigliani, Merton H. Miller, The Cost of Capital, Corporation Finance and the Theory of Investment. The American Economic Review, 1958, 48-3:261-297.

[2] D.K.Y Abeywardhana, Capital Structure Theory: An Overview. Accounting and Finance Research, 2017, 6(1): 133-138.

[3] Murray Z.Frank, Vidhan K. Goyal, Trade-off and Pecking Order Theories of Debt, Handbook of Corporate Finance: Empirical Corporate Finance, 2008.

[4] Bradley, M., Jarrell, G., Kim, E.H., On the Existence of An Optimal Capital Structure: Theory and Evidence. Journal of Finance, 1984, 39:857-78

[5] Kraus, A., Litzenberqer, R.H., A State-Preference Model of Optimal Financial Leverage. Journal of Finance, 1973, 33:911-922.

ISSN:2790-1661 DOI: 10.56028/aemr.3.1.303

[6] Strebulaev, I.A., Do Tests of Capital Structure Theory Mean What They Say?. Journal of Finance, 2007, 62-4:1747-1787.

[7] Hennessy, C.A., Whited, T.A., Debt Dynamics. Journal of Finance, 2005, 60:1129-1165.

[8] Cadsby, C.B., Frank, M.Z., Maksimovic, V., Pooling, Separating, and Semi-Separating Equilibria in Financial Markets: Some Experimental Evidence. Review of Financial Studies, 1990, 3:315-342.

[9] Eckbo, B. Espen, Norli, Oyvind, The Choice of Seasoned-Equity Selling Mechanism: Theory and Evidence. Tuck School of Business, Working Paper, 2004, 2004-15. Available at: https://ssrn.com/abstract=635649 or http://dx.doi.org/10.2139/ssrn.635649.

[10] Baker, M., & J. Wurgler, (2002), Market timing and capital structure, Journal of Finance, 57. https://doi.org/10.1111/1540-6261.00414

[11] Haohan Dong, Rong Zhang, Bojie Xie, The Impacts of Market Timing on Capital Structure. Journal of Coastal Research, Special Issue No. 106: Advances in Coastal Research: Engineering, Industry, Economy, and Sustainable Development (Summer 2020), 2020: 329-333.

[12] Yu shang, Ampirical Study of EVA on Capital Structure-based on New Energy Shipping Companies Data. Journal of Coastal Research. Special Issue 83: Advances in Sustainable Port and Ocean Engineering (FALL 2018). 2018: 828-832.

[13] Douglas Ayres, Steven Dolvin, Determining Optimal Capital Structure: An Empirical Example. Journal of Financial Education. 2019, 45(2): 243-257.

[14] Brennan, M.J., Schwartz, E.S., Optimal Financial Policy and Firm Valuation. Journal of Finance, 1984, 39:593-607.

[15] Leland, Hayne, Risky Debt, Bond Covenants and Optimal Capital Structure. Journal of Finance, 1994, 49:1213-1252.

[16] Leland, Hayne, Klaus Toft, Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads. Journal of Finance, 1996, 51:987-1019.

[17] Titman, Sheridan, Sergey Tsyplakov, A Dynamic Model of Optimal Capital Structure. Review of Finance, 2007, 11-3:401-451

[18] Joshua D. Rauh, Amir Sufi, Capital Strucuture and Debt Structure. The Review of Financial Studies, 2012, 23-12:4242-4280.

[19] Sami Attaoui, Patrice Poncet, Capital Structure and Debt Priority. Financial Management, 2013, 42-4:737-775.

[20] Nengjiu Ju, Hui Ou-Yang, Capital Structure, Debt Maturity, and Stochastic Interest Rates. The Journal of Business, 2006, 79-5:2469- 2502.

[21] Timmer, Jan H., Understanding the Fed Model, Capital Structure, and Then Some. Working Paper, 2012. Available at: https://ssrn.com/abstract=1322703 or http://dx.doi.org/10.2139/ssrn.1322703.

[22] Vasicek, O., An Equilibrium Characterization of the Term Structure. Journal of Financial Economics, 1977, 5: 177-188.

[23] Fang Haowen, European Option Pricing Formula Under Stochastic Interest Rate. Progress in Applied Mathematics, 2012, 4-1:14-21.

[24] Yang Xiaofeng, Yu Jinping, Li Shenghong, Albert Jerry Cristoforo, Yang Xiaohu,Pricing model of interest rate swap with a bilateral default risk. Journal of Computational and Applied Mathematics. 2010, 234:512-517.