Condorcet Theory: Exploration of Tie-breaking vote

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Abstract. Voting has played a crucial role in modern society, as it empowers individuals to voice their opinions and ensures representative and fair decision-making. However, no current voting algorithm can claim to be completely fair. In preferential voting systems, the Worst Defeat and Ranked Pair methods are often used to determine election outcomes. Due to fundamental differences in their principles, both algorithms exhibit distinct strengths and weaknesses, particularly regarding the weighting of tie-breaking votes. This paper delves into the specific cases of Worst Defeat and Ranked Pair in tie-breaking scenarios, offering an integrated analysis and proof of related issues. Furthermore, it identifies the most suitable contexts for each algorithm, aiming to provide general recommendations for the usage of voting algorithms across various situations.

Keywords: Voting algorithms; Tie-breaking; Preferential Ballot Election.

1. Introduction

From political elections to workplace decisions and even daily trivial matters, voting is an important method for producing representative results. A common issue in most voting processes is the occurrence of ties, which can create additional complications and disputes. In such cases, tie-breaking vote that ensures fairness without affecting the regular voting outcome is introduced. The tie-breaking vote is typically either randomly generated by the system or voluntarily created before voting, and by being less weighted, it ensures fairness in regular scenarios while being crucial in resolving ties.

As the core to support the voting process, voting algorithms have been continually studied and optimized. In recent years, numerous voting algorithms have emerged based on the theories of Borda and Condorcet, enhancing both their conceptual frameworks and rigor. However, every algorithm has its drawbacks. For instance, the ”Plurality” voting system used by the majority of U.S. states often elects representatives who do not reflect the preferences of the entire population (Silver, 2017). Similarly, the preferential voting system used in the 2013 mayoral election in Minneapolis introduced another issue: a candidate’s poor performance could paradoxically aid their victory. Obviously, such outcomes are not desirable in a voting system.

To address these issues, various improved algorithms have been developed. The Worst Defeat and Ranked Pair voting algorithms, known for their high efficiency and accuracy, have been widely used in preferential voting. In preferential voting, each voter ranks candidates by their preference, displaying a relationship of victory between them. However, in many cases, the tie-breaking vote, which is only meant to break the tie, may unexpectedly become more advantageous than a regular vote. By examining the conditions and causes of such instances,

this paper demonstrates why the tie-breaking vote is usually at a disadvantage. At the same time, this paper explores the counterexamples that being a tie-breaking vote does become advantageous, revealing certain flaws in specific voting algorithms and providing corresponding advice.

2. Literature Review

Preferential voting has endured a long history of development. In the preferential vote, unlike systems where voters can only vote for a single candidate, voters can express preferences for multiple candidates. In this way, a more accurate and reliable reflection of voter intent and results is provided. However, this democratic approach also increases the complexity of voting algorithms, as the outcomes represent more than just numbers—they convey relationships. Thus, various voting algorithms have been invented.

Two major schools of thought dominate the voting algorithms: the Borda and Condorcet methods. In 1770, French mathematician and political scientist Jean-Charles de Borda proposed a voting algorithm that allocates votes based on the rank order of preferences provided by each voter (Emerson, 2013). The method emphasizes a candidate's overall support and is widely used in point-based competitions like sports. However, the Borda count is susceptible to manipulation and, therefore, can lead to distorted results.

Around the same time in the 18th century, French mathematician and philosopher Marquis de Condorcet introduced a method focused on finding a candidate who would stand out from others in most, or even all, pairwise comparisons, making the election outcome more compelling (Fishburn, 1977). The core of the Condorcet method is to determine a comparative relationship or inequality between candidates. By focusing on pairwise victories rather than vote totals, Condorcet's approach went a long way toward preventing the possibility of rigging elections (Tideman, 1987). Due to its strong resistance to strategies and reliability, it's commonly used in academic and political elections (Brams & Fishburn, 2007). However, the method is more complex and prone to a phenomenon known as the Condorcet paradox, where collective preference cycles may prevent the identification of a clear winner, complicating the voting process further.

As voting theory evolved, various derivative algorithms emerged. Developments in Condorcet theory have largely focused on addressing the Condorcet paradox. Ranked Pair is a Condorcet's method that sorts pairwise matchups and locks in the relationship with the strongest amount of votes first (Parkes & Xia, 2012). Another notable method, Worst Defeat (also known as the Minimax method), is a refinement of Condorcet method. Developed further in the 20th century, it has become a common tool for handling scenarios without a Condorcet winner. Worst Defeat is now used in systems that aim to minimize the impact of strategic voting and ensure fairness, and it is often analyzed in comparison to more complex Condorcet methods like the Schulze method or Ranked Pairs.

Despite advancements in voting algorithms aimed at resolving inherent challenges, one critical aspect of preferential voting has been overlooked: the tie-breaking vote. The voter to be the tie-breaking vote is either voluntary or randomly generated by the system. The tie-breaking vote has a lower weight while preserving the order of voter's preference. At the same time, it does not disrupt the normal outcome of the election. Tie-breaking votes were introduced to address issues such as ties in all voting algorithms, including the Condorcet paradox. However, in some cases, these tie-breaking votes can lead to counterintuitive outcomes in which they hold the upper hand. This paper aims to investigate and prove phenomena related to tie-breaking votes, highlights an overlooked aspect of voting systems, and offers recommendations for the application of these methods in contemporary voting scenarios.

3. The Matrix of Victory

In a preferential ballot election, voters have to rank all the candidates. In other words, voters voters rank candidates in order of their preference. For example: a voter votes "B C D A" . This indicates that this voter prefer candidate B the most and ranks candidate A the lowest.

To collect and record the results and preferences, the Matrix of Victory (MOV) is used. The previous voter, for example, has a MOV: In any one-on-one match-up, a candidate gets '1' in its row against the column of its opponent if it's ranked higher than its opponent. Respectively, it will get a '-1' if it loses. The MOV of "B C D A" is shown in Figure 1. Candidate B is ranked higher than any candidate else, so it has three 1s in row.

All the single MOVs will add up to a final MOV which serves as a means for the voting algorithms the paper will present later to determine the winner of a preferential ballot election.

	A	B	C	D
A	0	-1	-1	-1
B	1	0	1	1
C	1	-1	0	1
D	1	-1	-1	0

Figure 1: MOV of ”B C D A”

4. The method of determining the winner of a Preferential Ballot Election

There are plenty of methods or algorithms to determine the winner of a preferential ballot election, such as plurality, instant runoff voting, the Borda Count, and so on. This paper will mainly focus on the Worst Defeat and the Ranked Pairs algorithms, a quite convenient method among a wealth of candidates and a quite logical method respectively.

4.1 Ranked Pair

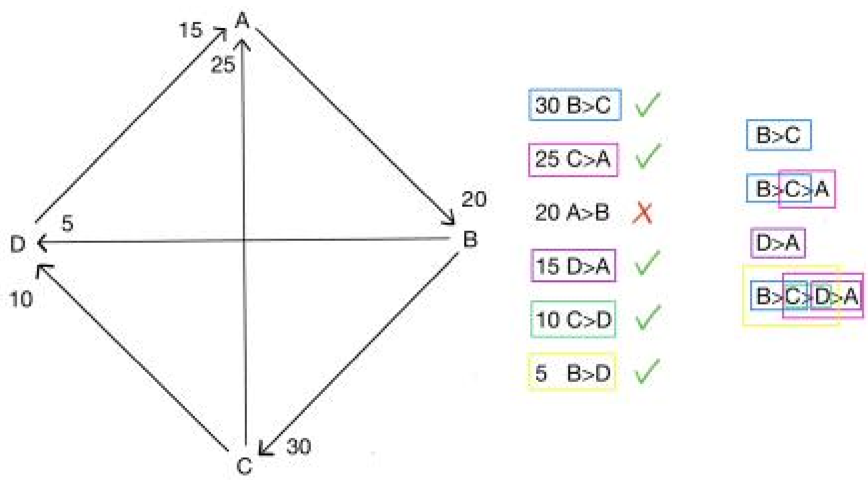

Figure 2: Ranked Pair Operation

In Ranked Pair, the most convincing ”Beating Chain” based on votes is expected to be found. Take the MOV in Figure 2 as an example. The votes in one-on-one matches, voting is selected and ranked based on relationship: [30, B > C], [25, C > A], [20, A > B], [15, D > A], [10, C > D], [5, B > D] In this case, B > C will become the main chain, and the later relations, according to the order, without contradicting the main chain can be accepted. For example, when it comes to the relation A > B, it contradicts the current chain of B >C > A, so it is ignored. The final Beating Chain would be B > C > D > A, so the winner is B.

4.2 Worst Defeat

“Worst Defeat” is a method used to determine the winner by selecting the candidate whose most significant loss in a one-on-one match-up is the smallest. Figure 3 here is a MOV example. According to the Figure, candidate D wins the election by having the least Worst Defeat.

	A	B	C	D	Worst Defeat
A	0	20	-25	-15	-25
B	-20	0	30	5	-20
C	25	-30	0	10	-30
D	15	-5	-10	0	-10

Figure 3: MOV with Worst Defeat

One might notice that the winners given by these two methods are different. This is due to the focuses of these two algorithms are different. Ranked Pair aims to find the most convincing winning chain, while the Worst Defeat focuses more on the setbacks. With three or less candidates, these two methods would give the same winner since the relationships in those situations are simple.

5. Tie-breaking

Both methods will face a tie. When there is a tie, the Worst Defeat method might not be able to determine the candidate with least significant loss, and Ranked Pairs method might not be able to give a chain due to some equivalent and contradictory relations. Therefore, a tie-breaking vote is needed in advance of a voting event.

5.1 Tie-breaking vote convention

A tie-breaking vote should possess several properties. It has to address the tie problem. At the same time, it cannot affect the normal election result. Therefore, its weight must be adjusted. Usually, the tie-breaking vote follows this convention:

For n candidates, $\frac{n(n-1)}{2}$ one-on-one match-ups are obtained as $s(n)$. Let $d(n)$ be the number of digits of s(n).

For the top priority candidate of a voter, it beats the least preferred candidate by s(n) × 10−d(n) , the penultimate preference by {s(n)−1}×10−d(n), and the second preference by {s(n)−(n − 2)}× 10−d(n) .

For the second preference, it beats the least preferred candidate by {s(n)−(n −1)}× 10−d(n) , the penultimate preference by {s(n)−n}×10−d(n), and the third preference by {s(n)−2(n − 2)}× 10−d(n) .

This continues until the penultimate preference beats the least preference by 1 × 10−d(n) .

In Figure 3, for instance, s(n) is $\frac{4(4-1)}{2}$ which is 6. Number 6 only has one digit, so $d(n)$ here is 1. Start from the top priority candidate, it beats the least preferred candidate by 6 × 10−1 which is 0.6, the penultimate preference by (6 − 1) × 10−1 which is 0.5, and the second preference by 6 −(4 −2) × 10−1 which is 0.4. The second preference beats the least preferred candidate by 6 −(4 −1) × 10−1 which is 0.3. The penultimate preference and the third preference here are the same person, so the second preferred candidate beats this person (6 −4) × 10−1 or 6 −2(4 −2) × 10−1 which is 0.2.

In this way, the preference of the voter is ensured. Also, all the match-ups weight differently. Figure 4 shows a 4-candidate situation as an example.

	A	B	C	D
A	0	0.4	0.5	0.6
B	-0.4	0	0.2	0.3
C	-0.5	-0.2	0	0.1
D	-0.6	-0.3	-0.1	0

Preference:
A
B
C
D

Figure 4: MOV of traditional tie-breaking vote

6. Is being a tie-breaking vote always a disadvantage?

According to Figure 4, voting as a tiebreaker is actually a disadvantage because voters have a lower vote weight than regular voting.

To think of a counter-example, a scenario is held: the election with 2 more votes to go. At this moment, if voter A becomes the tie-breaking vote, voter A's top preference wins the ballot. If voter B becomes the tie-breaking vote, voter B's top preference wins the ballot instead.

6.1 Construction through Worst Defeat

Intentionally, the problem in the worst-fail algorithm is a relatively simple construct. Here is the scenario: the current Worst Defeat situation is A loses D by 6, B loses E by 6, and C loses F by 4. While, the candidates D, E, F have enormous worst defeat themselves (see Figure 5). Two votes left are [A D B E F C] by voter A and [B E A D F C] by voter B. Either of them can turn the Worst Defeat situation into a tie as a regular vote, which is A loses D by 5, B loses E by 5, and C loses F by 5. Now, since the voter left will become the tie-breaking vote with the top preference weighting the most, the top preference of this voter wins by Worst Defeat. In this case, both of the voters want to be the tie-breaking vote.

This example is constructed by grouping the candidates with the candidates who give them the

	A	B	C	D	E	F
A	0	1	-3	-6	20	22
B	-1	0	2	23	-6	21
C	3	-2	0	24	25	-4
D	6	-23	-24	0	-10	12
E	-20	6	-25	10	0	-11
F	-22	-21	4	-12	11	0

Figure 5: 6-candidates Situation counter-example

Worst Defeat. There are three groups in this 6-candidate case which are AD, BE, and CF. Similarly, such examples can be constructed in 5-candidates and 4-candidates situations with two groups(5-candidates one has a group of three).

Theorem 1

In the case of three candidates, the favorable circumstances of a tiebreaker do not exist.

Proof

Three-candidates situation has a clear logic and a simple structure, making the operating process easy for voting algorithms.

In Ranked Pair, a winning relationship has to be determined. For other situations, various paths and routes connecting every single candidate need to be considered, while a few arrows of relation are shown. The dominant state shown in Figure 6 and the cyclic state shown in Figure 7 are two common situations. In the dominant state where a Condorcet winner

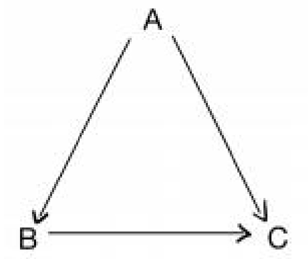

Figure 6: Dominant Situation

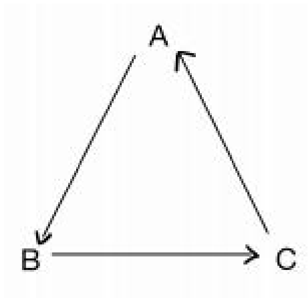

Figure 7: Cyclic Situation

exists, the tiebreaker vote has no effect on the result, as it cannot equal or reverse arrows with weights less than 1. The winner of Condorcet always wins the game because the three arrows are logically aligned. In the circular case, this relationship is also easy to determine. You only need to consider the two largest arrows and then determine the relationship, because the last arrow will not follow logic. In other words, only the arrows with the least intensity need to be ignored. Therefore, normal voting is preferred because it has more weight than a tie vote to increase the outward arrow or decrease the inward arrow.

In Worst Defeat, the Condorcet winner in the Dominant situation or a candidate who loses the least wins. Still, the tie-breaking vote cannot change the outcome in the dominant state. In the Cyclic state, the candidate with the least to lose is the candidate with the least inward arrow. For the same reason, the tiebreaker vote gives less weight to the inward arrow.

Due to three-candidate situation's simplicity, a tie-breaking vote won't get any advantage.

6.2 Ranked Pair Issue

When it comes to the Ranked Pair, the previous examples may not stand. When all the results of one-on-one match-ups are listed, the Beating Chain is shaped by those large votes.

7. Conclusion

The Rank Pair algorithm strictly follows Cordorcet's theorem, as long as there is a Condorcet winner, the rank Pair can always pick one. The result provided by Ranked Pair is clear and convincing. However, the overall structure might be impacted and the process may be more complicated and time-consuming by slight differences in vote count. Ranked Pair can construct a stable ranked relationship, making it suitable for scenarios where ranking and credibility are highly valued. The main feature of Worst Defeat is its efficiency, with a simple and lucid process. Moreover, the results that emerge from the worst failures are often not extreme or unreasonable. However, this approach tends to favor "compromise" candidates and does not adequately reflect public opinion. The results are also less intuitive and may be misunderstood. Worst Defeat is more suitable for scenarios requiring quick decisions and a reasonable number of candidates.

To address ties and the Condorcet paradox, the tie-breaking vote is used with a small weight to break the deadlock. However, current voting methods are not perfect, and there are some flaws in the study of tie-breaking votes. This paper reveals some counterintuitive cases, where tie-breaking votes have more advantages than regular votes, through examples constructed utilizing graphs and the characteristics of voting algorithms. Nevertheless, this phenomenon was only demonstrated through counterexamples, without systematic proof. In the future, more rigorous methods like Ranked Pairs method can be explored for further research on tie-breaking votes. At the same time, more detailed and improved voting algorithms or tiebreaker voting schemes should also be studied. That is how real democracy can be achieved.

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