Kelly's criterion for optimization

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Abstract. Since first proposed by John Larry Kelly, the Kelly criterion has become one of the most reliable methods to determine the maximum profit strategy in investment and gambling. In this work, the basic principle of Kelly's criterion is first reviewed. Then I discuss its application in the irrelative, un-correlated, and anti-correlated investment scenario. Finally, some possible follow-up studies and extensions of Kelly's criterion are prospected.

Keywords: Kelly's criterion, optimization for investment, Central limit theorem, probability.

1. Introduction

In this era, due to the rapid development of the economy and the driving force in the market, there are many investment projects for people to choose from. The diversity of investment mode and returns makes it hard for people to decide. How to maximize the interests of investors, namely, how to return the money spent on their investment to the maximum profit is an important issue for finance. This has become a research hotspot for many financiers, entrepreneurs and ordinary investors.

The Kelly formula was originally developed by John Larry Kelly [1], a member of AT&T Bell Laboratories, based on the work of his colleague Claude Ellwood Shannon on the noise of long-distance telephone lines [2]. Kelly explains how Shannon's information theory could be applied to the problem of a gambler with inside information when betting on horses. Due to the primitivity of communication technology at that time, there was a time difference in the broadcast between the west and east of the United States, so people on the East Coast would send the news to gamblers on the West Coast to help them place bets to win the horse race. But because of backward communication technology, the information a gambler receives is not always that accurate, but the gambler wants to determine the best bet. For this, Kelly tried to find a way to give him a useful advantage on the condition that the information he receives does not need to be completely accurate. As a result, Kelly developed a theory of how many bets to make to maximize profits in such uncertain circumstances.

In this paper, I will discuss how to use Kelly's criterion and formulas to optimize investor returns. Firstly, some mathematical formulas are recalled. Then the optimization in the irrelative, uncorrelated, and anti-correlated occasions are discussed.

2. Interpretation of symbols

A_n is the money which possessing after n bets
A_0 is the money which initially has
x is the fraction of the bankroll to be wagered
b is the current odds
w is the number of winning bets
is the numbers of losing bets
B_n is the current value
B_0 is the initial value
G is the compounding growth rate
M is the participant's bankroll
M' is new bankroll
x is the fraction of the money in the bankroll

3. Preliminaries

(1) Concept: Concept of compounding growth rate *G* [3]

G represents the increasing or decreasing status of an investment in n years or in n turns.

$$G = (\frac{B_n}{B_0})^{\frac{1}{n}} - 1$$

where B_n is the current value, B_0 is the initial value.

(2) Definition: Happiness of wealth (expectation) [3]

Happiness is a value to measure people's expectations (logw) in Kelly's criterion.

$\text{Happiness(H)} = \log(w)$, where w is money or wealth.

(3) Theorem: Central limit theorem [4]

For independent random variables $x_1, x_2, x_3, \ldots, x_n, \ldots$, their math expectation and variance are denoted as $E_{(x_i)} = \mu, D_{(x_i)} = \sigma^2$, where $i = 1, 2, 3, \ldots$

For random x, if the distribution function $F_n(x) = P\{\frac{\sum_{i=1}^{n} x_i - n\mu}{\sigma\sqrt{n}} \le x\}$ satisfies $\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} P\{\frac{\sum_{i=1}^{n} x_i - n\mu}{\sigma\sqrt{n}} \le x\}$, the distribution is a Gaussian distribution.

(4) Theorem: Kelly's formula [5]

To make our profit maximum in the game, we should invest the fraction of total bankroll of $x = \frac{pb-q}{b}$. That is when the fraction of the money we invest is $\frac{pb-q}{b}$, maximum average per bet rate of money return can occur (Verified in chapter 7).

4. Optimization for irrelative investment scenarios (A coin for a bet)

To help understand this scenario, let us start the discussion with a simple game as an example. The first game is flipping each coin for each bet in one game. The two different bets will not have any influence on each other. Namely, every bet here is independent.

(1) Triple or nothing

Assuming a person who has money of *M*. He bets money *x* on the game, then flips one coin. If it is a head, the participant will win 3*x*; Otherwise, the participant is going to forfeit the money he bets. To vividly illustrate this scenario, we assume the person's bankroll is 100 dollars. At each turn of the game, he bet half of the bankroll.

$$\text{Head:} \quad M = 100 - \frac{1}{2} \times 100 + 3 \times 50 = 50 + 150 = 200$$

$$\text{Tail:} \ 100 - \frac{1}{2} \times 100 + 0 = 50$$

Substituting the money by symbol, we have:

$$\text{If it is head: } M \to M \cdot (1 - x + 3x) \to M \cdot (1 + 2x)$$

$$\text{If it is tail: } M \to M \cdot (1 - x + 0x) \to M \cdot (1 - x)$$

Thus, the new bankroll (*M*') of the person can be denoted by:

$$\text{Head: } M' = 2M$$

$$\text{Tail:} M' = \frac{1}{2} M$$

Presuming the person played the game 10 times and got: head, tail, head, head, tail, head, tail, head tail, head. The bankroll M' is going to be $M' = M \cdot 2 \times \frac{1}{2} \times 2 \times 2 \times \frac{1}{2} \times 2 \times \frac{1}{2} \times 2 \times \frac{1}{2} \times 2 = 4M \Longrightarrow M' = 4M$. However, due to the multiplicity nature, the central limit theorem cannot be applied in this case. To avoid this problem, a logarithm for the expectation of profit can be used. Researchers found that the logarithm curve is more suitable for denoting investing behaviors. In statistics, investors always need to monitor some investment trend, so they need to observe past behaviors, sometimes they use the central limit theorem, but this theorem is only capable of addition, while logarithms solve it perfectly.

As a result, for simplicity of algorithm and expression of people's happiness. It is better to use the logarithm to express expectation since logarithm is possible to change multiplicity to addition, and exponential to multiplicity. Expressing the person's expectation of logarithm is:

$$log(M) = H$$

$$H \to H + \log(1 + 2x)$$

$$H \to H + \log(1 - x), \text{ respectively.}$$

In this case, the new expectation can be written as

$$\text{Head: } H' = H + log2$$

$$\text{Tail: } H' = H - log2$$

So, after 6 turns (head, tail, tail, head, head, head), we have new expectation H':

$$H + log2 - log2 + log2 + log2 - log2 + log2 - log2 + log2 - log2 + log2 = H + 2log2$$

Here, each observation is sharing an additional relation with others. The central limit theorem applies!

In this scenario, people flip a coin to determine a win or lose, which has the same possibility. Therefore, we can simulate it as the participant always get a head and a tail in 2 turns. The bankroll will be $M' = M(1 + 2x)(1 - x)$. To make the maximum profit in the game, we need to maximize $(1 + 2x)(1 - x)$. For function $f_{(x)} = (1 + 2x)(1 - x)$, easy to know the critical point is $x = \frac{1}{4}$ (see Figure 1). And $f_{(\frac{1}{4})} = \left(1 + 2 \times \frac{1}{4}\right)\left(1 - \frac{1}{4}\right) = 3 \times \frac{3}{4} = \frac{9}{8}$, so new bankroll is $M' = \frac{9}{8}M$ in each 2 turns.

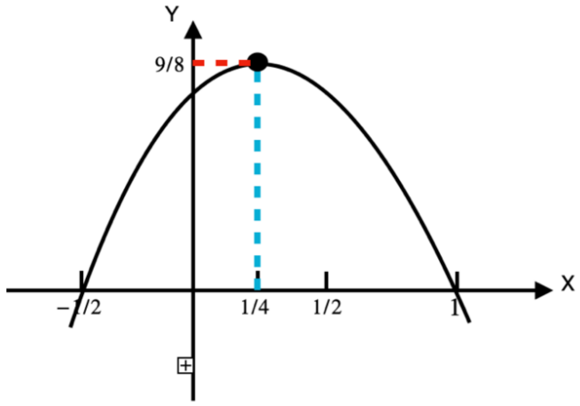

Fig. 1 Plot of function $f_{(x)} = (1 + 2x)(1 - x)$, and its maximum.

It shows that if we bet a quarter of our bankroll, we can optimize the profit, and we gain more if play more!

(2) Double or nothing

Same as triple or nothing in (1), if a person plays the game with his bankroll M, we have:

$$\text{If it is head: } M \to M \cdot (1 - x + 2x) \to M \cdot (1 + x)$$

$$\text{If it is tail: } M \to M \cdot (1 - x + 0x) \to M \cdot (1 - x)$$

Also, discuss the flipping coin case, we have half-half probability, we assume that the person has a win and a loss in each 2 turns. It will be $M' = M(1+x)(1-x)$, where M' is the new bankroll. Define $g_{(x)} = (1+x)(1-x)$, easy to get critical point is 0, $g_{(0)} = 1$, while the curve is concaving down. It means, if he bets money of x in the game double or nothing, he is going to surely lose money! To verify this, if bet half of bankroll in the game, after 2 turns, supposing a win and a loss:

$$M' = M \cdot \left(\frac{1}{2} + \frac{1}{2} \times 2\right)\left(\frac{1}{2} + 0\right) = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4} M < M.$$

In this scenario, it demonstrates whatever people bet, they will definitely lose money if they play as many turns as possible. The best choice to optimize profit is quitting!

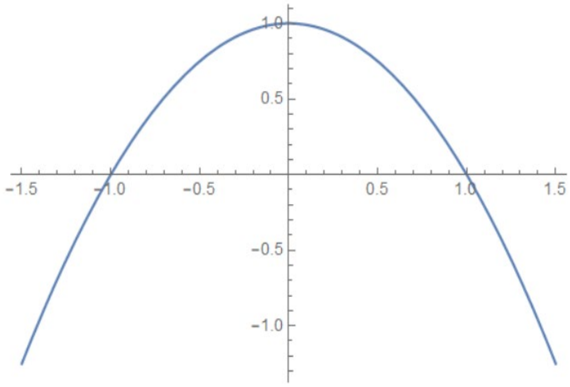

Fig. 2 Plot of function $g_{(x)} = (1+x)(1-x)$

5. Distribution of uncorrelated investment optimization (Two coin for two bets)

Next, we will discuss optimization of the uncorrelated investment which means in one game we flip 2 coins simultaneously, and we give the investment strategies for the case.

Suppose we change the rules, now we have 2 games to bet, one is **triple or nothing,** and another is **quadruple or nothing**. Now, we are going to discuss how to allocate our bankroll to bet in order for sake of maximum return. Firstly, let us discuss a simple example:

Assuming a person has M dollars in his bankroll, and there are 2 games for him to bet: Bets money x into the games, and then flips a coin, for the first one if wins, he will get 3x, loses to forfeit the bet (triple or nothing), another is wining to get 4x, lose to lose the bet (quadruple or nothing). We can easily see 2 games are uncorrelated. Due to those 2 have no impact on each other, we call them independent. so, we are still going to bet a quarter of bankroll to the first game, use the same method in the last scenario, see $f_{(x)} = (1-x)(1+3x)$, critical point is $x = \frac{1}{3}$, $f_{\left(\frac{1}{3}\right)} = \frac{2}{3} \times 2 \cdot M = \frac{4}{3} M$, it profits. Because of this, betting $\frac{1}{3}$ of the bankroll is the wisest choice.

In conclusion, for these 2 games, best choice to distribute is:

$$M = \frac{1}{4} M\ (triple\ or\ nothing) + \frac{1}{3} M\ (quadruple\ or\ nothing\ ) + \frac{5}{12} M\ (\ cash).$$

6. Optimization for anti-correlated investment scenarios (A coin for two bets)

The discussions above include two normal cases. However, another anti-corrleated investment exsit, which will be disscused in this chapter.

(1) Two 'good' bets

What if the game rules change to heads for triple or nothing, tails for quadruple or nothing? We are going to discuss the case in this section. Let us in the same way to start with an example.

Suppose a person has 100 dollars, he is going to take part in the game above, how to allocate his money if he wants to make the most profit?

Let *x* be the fraction of his bankroll to bet on the triple game, and y for the quadruple game, so that cash remains 1-*x*-*y*.

If head: $M' = M(1 - x - y + 3x + 0y)$

If tail: $M' = M(1 - x - y + 0x + 4y)$

Table 1 Outcome of the two 'good' bets.

	Bet 1	Bet2
head	3x	0
tail	0	4y

Flipping a coin can always give a half-half probability when the game is repeated over again and again. Thus, suppose it as a head and a tail in each 2 turns:

$$M' = M(1 + 2x - y)(1 - x + 3y)$$

To maximize *M*', we construct a function as $h_{(x,y)} = (1 + 2x - y)(1 - x + 3y)$, a two-variable function, where $x \geq 0, y \geq 0, x + y \leq 1$. Solve the function, let $z = (1 + 2x - y)(1 - x + 3y) \rightarrow z = 1 + 2y + x - 2x^2 + 7xy - 3y^2$. To get the stationary point, we take partial derivative and let them be 0: $f_x = 1 - 4x + 7y = 0, f_y = 2 + 7x - 6y = 0$. Solve equations, it yields negative result, so the stationary point is on the line of $x + y = 1$.

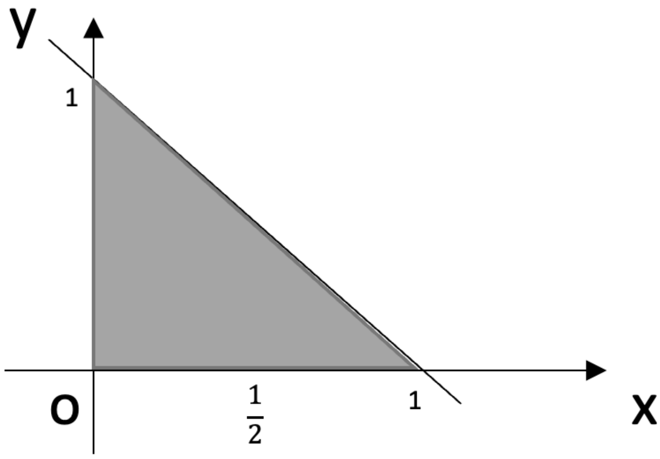

Fig. 3 Plot of $x + y = 1$

And it is actually at $\left(\frac{1}{2}, \frac{1}{2}\right)$ by calculated. See Figure 2 and Figure 3, they also illustrate $\left(\frac{1}{2}, \frac{1}{2}\right)$ leading to the maximum value for the function.

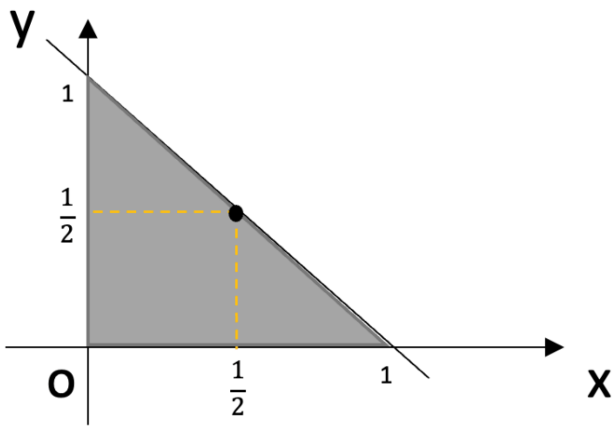

Fig. 4 Plot of boundary $x + y = 1$ for $h_{(x,y)} = (1 + 2x - y)(1 - x + 3y)$

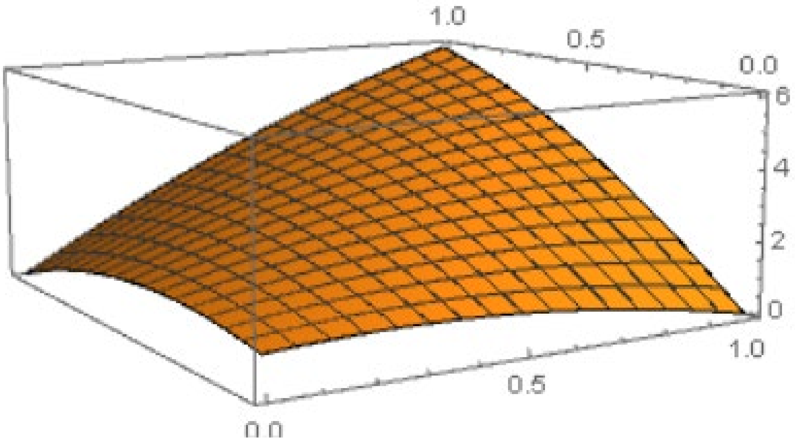

Fig. 5 Plot of $h_{(x,y)} = (1 + 2x - y)(1 - x + 3y)$

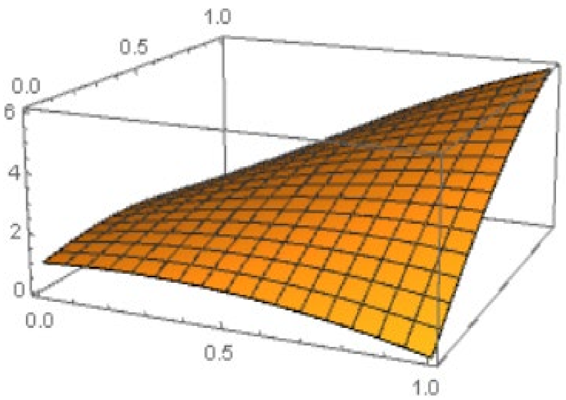

Fig. 6 Plot of $h_{(x,y)} = (1 + 2x - y)(1 - x + 3y)$ in another view

From now on, we see it is reasonable to bet a half of the bankroll to the triple game, at the same time, bet another half to the quadruple game, it yields the best return. i.e.,

$$M = \frac{1}{2}M\ (for\ game\ 1) + \frac{1}{2}M\ (for\ game\ 2) + 0\ (remain\ for\ cash).$$

We know the probability of wins and losses is $\frac{1}{2}$, as we explained before. As a result, the game can be written below:

$$\text{If head: } M' = M \cdot \frac{1}{2} \times 3 = \frac{3}{2}M$$

$$\text{If tail: } M' = M \cdot \frac{1}{2} \times 4 = 2M$$

So, every 2 turns of 1 win and 1 loss, we have $M' = M \cdot \frac{3}{2} \times 2 = 3M$. The participant is going win money by continuing to play.

(2) A 'good' and a 'bad' bets

This time, suppose we have 2 bets of 1 good and 1 bad for the game. In detail: The first game is triple or nothing, which means if he gets a head he wins 3 times his bet, if tail, he is going to get only 1.5 times his bet. In this case, how to make the best profit?

Table 2 Outcome of 'good' and a 'bad' bets.

	Bet 1	Bet2
head	3x	0
tail	0	1.5y

Similarly, we still assume he plays the game many times, and the probability behavior is going to be half and half. It shows a win and a loss in every 2 turns. Then, constructing the profit function like last example: $w_{(x,y)} = (1 + 2x - y)(1 - x + 0.5y)$, where $x \geq 0, y \geq 0, x + y \leq 1$. Maximize $w_{(x,y)}$, by calculating and getting the extreme point which is also $\left(\frac{1}{2}, \frac{1}{2}\right)$. Verifying by draw the plot by mathematica, also shows that (see Figures 4, 5).

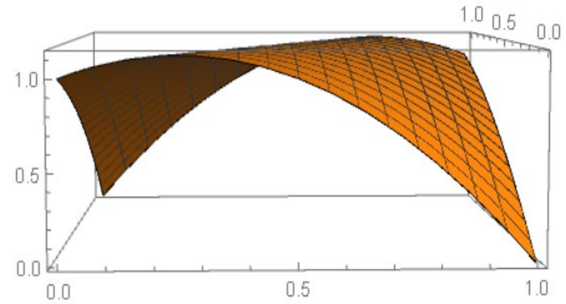

Fig. 7 Plot of $w_{(x,y)} = (1 + 2x - y)(1 - x + 0.5y)$

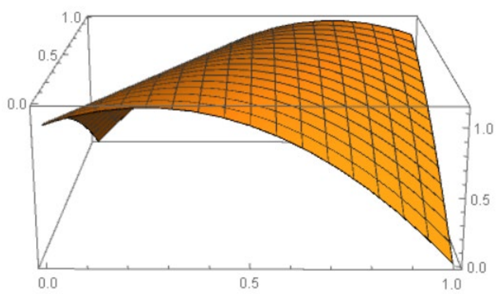

Fig. 8 Plot of $w_{(x,y)} = (1 + 2x - y)(1 - x + 0.5y)$ in another view

The function takes the extreme value in the boundary of $x \geq 0, y \geq 0, x+y \leq 1$, which is at $\left(\frac{1}{2}, \frac{1}{2}\right)$. Evaluating the profit by every 2 turns, see $M' = M \cdot \frac{3}{2} \times \frac{3}{4} = \frac{9}{8} M$. Thus, obviously, the participant is going win more money if he plays as possible as he can, if he bet half and half!

(3) Conclusion for the scenario

As we saw in the upper examples, in the anti-correlated scenarios, for 2 bets in a game, let us discuss the general case for gamblers what to do if they want to gain wealth. Assuming a person who has money of M in his bankroll, and he is going to bet x for bet1 and y for bet2 (see table 3).

Table 3 outcomes for anti-correlated bets

	Bet 1	Bet2
head	ax	0
tail	0	by

Let $1 \leq a \leq b$, since the gamblers have the opportunities to gain money, then they would join. In this case, we write the profit function as $H_{(x,y)} = (1-x-y+ax)(1-x-y+by)$, where the boundary is $x \geq 0, y \geq 0, x+y \leq 1$ (see Figure 3).

However, what values are a and b taken, are directly affecting the strategy to bet.

i. **Case 1: If $1 \leq a \leq b \leq 2$, do not bet.**

We have already discussed the case in chapter 4, part (2), the double or nothing case. In this case, whatever people optimize their plan to bet, they will lose money after many turns. For a, b less than 2, it is equivalent to scaling the plot (see Figure 9).

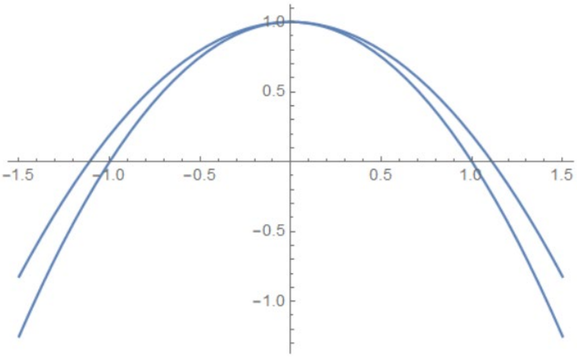

Fig. 9 Plot of $g_{1(x)} = (1+x)(1-x)$ and $g_{2(x)} = (1-0.9x)(1+0.9x)$

It is clear that when x is bigger than 0, the value of the profit function will be less than 1, which means losing money! To conclude this, we say if $1 \leq a \leq b \leq 2$, it is wise to quit the game. Because of joining is losing!

ii. If $\frac{1}{a} + \frac{1}{b} \geq 1$, $2 < b$, bet b with suitable fraction of bankroll.

Here $b > 2 \rightarrow \frac{1}{b} < \frac{1}{2}$, then if wants to let $\frac{1}{a} + \frac{1}{b} \geq 1$, $\frac{1}{a} > \frac{1}{2} \rightarrow a < 2$. Knowing that if the bet rate is less than 2, we will not be going to choose to bet on that a. Thus, it is sure for betting money on b, since $2 < b$, this bet will help us gain money. The crucial step here is not only to bet on b but also need to think about what percentage to bet can maximize our profit, so using Kelly's criterion we have given previously, then we are all done!

iii. If $\frac{1}{a} + \frac{1}{b} < 1, 2 < b$, bet half and half of bankroll on each bet.

$b > 2 \rightarrow \frac{1}{b} < \frac{1}{2}$, so $\frac{1}{a} > \frac{1}{2} \rightarrow a > 2$. That is, for this scenario, both 2 bets of the rate a and b are bigger than 2. The profit function we have obtained as $H_{(x,y)} = (1 - x - y + ax)(1 - x - y + by)$ be bounded by $x \geq 0, y \geq 0, x + y \leq 1$, we have already solved it previously, that is at point $\left(\frac{1}{2}, \frac{1}{2}\right)$ yields the best profit. which means investing half and half of the money in the 2 bets can make money that returns the most (see Figure 4).

7. The strategy for more cases with non-half and half probability

Flipping coin games are much simpler than real investments or real gambling games. because we know a coin has 2 sides, if it is flipped, the results can only be head or tail, so the more times we play the probability is going to reach $\frac{1}{2}$ gradually. But we all know, the true rate cannot always be half, and even sometimes, like Kelly's problem, the information from others can be erroneous, caused by noise on the communication equipment. Even in some casinos, the dealer will not let you get access to true bet odds. Otherwise, you have your interline in the casino [6].

(1) Having interlines in the casino to invest (known true rate)

Suppose the case, we have our interlines. What should we invest or bet?

Suppose we have money of A_n initially. Then we are going to put money of $x \cdot A_0$ into the investment, then we obtain $x \cdot b \cdot A_0$ money if we ton in the first single bet. Counting in the original capital of investment we possess money of $A_0 \cdot (1 + bx)$ after one winning of bet. Similarly, if we lose a game, we will forfeit the mount of $A_0 \cdot (1 + x)$ money.

Following this regulation, after n bets of games, assuming we won w bets, and lost l bets, the money we possess should be $A_n = A_0(1 + bx)^w \cdot (1 - x)^l$, where n is the total number of bets, i.e. $n = w + l$.

This is the basic formula for the sort of games of bets. To maximize our profit in such games. The most vital mission is to ensure what percentage of our bankroll should be taken to invest.

Then to maximize the profit, it is equivalent to maximize the Compounding Growth Rate G of the game, then it is same to maximized $(\frac{B_n}{B_0})^{\frac{1}{n}} = [(1 + bx)^w \cdot (1 - x)^l]^{\frac{1}{n}}$.

$[(1 + bx)^w \cdot (1 - x)^l]^{\frac{1}{n}} = (1 + bx)^{\frac{w}{n}} \cdot (1 - x)^{\frac{l}{n}}$. Set $p = \frac{w}{n}$, $q = \frac{l}{n}$, and take the logarithm to convert multiplication to addition formulae, exponential to multiplication formulae.

i.e., $(1 + bx)^p \cdot (1 - x)^q = e^{\ln[(1+bx)^p \cdot (1-x)^q]} = e^{p \cdot ln(1+bx) + q \cdot ln(1-x)}$.

Construct a function $f_{(x)} = p \cdot ln(1 + bx) + q \cdot ln(1 - x)$.

$f_{(x)}{}' = \frac{pb}{1+bx} + \frac{-q}{1-x}$, let $f_{(x)}{}' = 0$ to obtain the critical point for the function. It yields that $x = \frac{pb-q}{b(p+q)} = \frac{pb-q}{b}$, since p and q are complementary events, i.e., $p + q = 1$.

Moreover, to ensure the critical point is the maximum. $f_{(x)}{}'' = \frac{-pb}{(1+bx)^2} + \frac{-q}{(1-x)^2}$, since $pb > 0$; $(1 + bx)^2 > 0$; q and $(1 - x)^2 > 0$. Thus, $f_{(x)}{}'' = \frac{-pb}{(1+bx)^2} + \frac{-q}{(1-x)^2} < 0$, it shows that the function of $f_{(x)}$ concaves down, means the critical point x is the maximum.

Thus, to make our profit maximum in the game, we should invest a fraction of the total bankroll of $x = \frac{pb-q}{b}$. That is when the fraction of the money we invest is $\frac{pb-q}{b}$, maximum average per bet rate of money return can occur. This is Kelly's formula [6, 7].

(2) No interline to invest

The main task here is almost done as we discuss how to invest in the upper case, the problem here is to know the true odds. Here we can use a theorem in statistics, the central limit theorem, see it in preliminaries.

Since we already defined happiness as expectation, see it in the preliminaries or chapter 4. After many times experiments, with the theorem we can finally get the true odds, the cons are that has to spend a lot to test [8].

8. Conclusion

In this paper, we first introduced how to use kelly's criterion for a simple game *flipping a coin* with half and half probability, we derived that in different scenarios we invest correspondingly as we calculated. To generalize the problem, in our life many situations do not have half and half probability, we used kelly's formula to give the best choice for gamblers or investors to determine. Even in many other cases, some mathematicians use the algorithm to grab the biz in blackjack in casinos. It is well known that kelly's optimization can still have many follow-up studies to improve human life.

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