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# Game theory- about The Game of Nim

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**Abstract.** Game theory has been widely used in economic life, and has had an important impact on economic development. This paper will analyze and explain some basic internal logic of game theory through some concrete cases related to the Game of Nim. It is found that the strategies of some theoretical models in game theory can be applied to the real game of chance, so these cases can be widely applied to actual economic life and social life.

**Keywords:** Game theory; The Game of Nim; winning strategy.

## 1. Introduction

The Game of Nim is a game where two players take stones from a pile of stones in multiple stacks (at least 2) and see who gets the last one to win. This essay is about the mathematical ways to win The Game of Nim. Also, there are several versions of The Game of Nim, one of which is called The Subtraction Game and is discussed below.

The possible contribution of this paper mainly has the following two aspects. Firstly, this paper takes The Game of Nim as a concrete case and enumerates a series of specific optimal strategies, which can serve as a reference for the ideas of game theory. Secondly, this paper finds that the optimal strategy of the theoretical model in game theory can be applied to the real game of chance.

## 2. The Original Version of The Game of Nim

As mentioned in the introduction, The Game of Nim is a game of which of the two competitors will take the last stone from several piles of stones when they are allowed to take any number of stones from only one pile in turn. Two players must decide who will start the game before it begins. To see how this game actually works, see a Table 1 here.

Table 1: Assuming there are two piles of stones with the numbers 3 and 6, A and B are playing this game. A starts.

| Number of stones on the first piles | Number of stones on the second piles | Steps of players |
|---|---|---|
| 3 | 6 | The game starts. |
| 2 | 6 | A starts, taking 1 stone from the first pile. |
| 2 | 1 | B starts, taking 5 stones from the second pile. |
| 0 | 1 | A takes 2 stones from the first pile. |
| 0 | 0 | B takes the last stones, and B wins. |

In the following example, B takes the last stone from the piles and wins the game, but is there a chance for A to beat B? This answer is yes. Here is the way A can win.

Table 2: Suppose there are two piles of stones with numbers 3 and 6, A and B play this game. A starts.

| Number of stones on the first piles | Number of stones on the second piles | Steps of players |
|---|---|---|
| 3 | 6 | The game starts. |
| 3 | 3 | A starts and takes 3 stone from the first |

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| | | |
|---|---|---|
| | | pile. |
| 3 | 1 | B is next and takes 2 stones from the second pile. |
| 1 | 1 | A takes 2 stones from the first pile. |
| 0 | 1 | B takes 1 stone from the first pile. |
| 0 | 0 | A takes the last stones from the second pile, and A wins. |

What is the difference between A's steps in Table 2 and 3? If we focus on the even number steps, we can find some clues.

Table 3: Suppose there are two piles of stones with numbers 3 and 6, A and B play this game. A starts.

| Number of stones on the first piles | Number of stones on the second piles | Steps of players |
|---|---|---|
| 3 | 6 | The game starts. |
| 3 | 3 | A starts and takes 3 stone from the first pile. |
| 3 | 1 | B is next and take 2 stones from the second pile. |
| 1 | 1 | A takes 2 stones from the first pile. |
| 0 | 1 | B takes 1 stone from the first pile. |
| 0 | 0 | A takes the last stones from the second pile, and A wins. |

We can see that each time A finishes taking stones, the number of stones in the two piles is the same. From this, we can conclude that A's winning strategy is to make the number of stones in both piles equal and follow B's steps. On the other hand, if the number of stones in the two piles is already equal, A should choose to follow B to win the game.

To make the game even more complicated, this time A and B play The Game of Nim with three piles of stones. In each pile there are 2, 5 and 6 stones. Should A choose the first move this time? What strategy should A use to win? Here is how A proceeds.

Table 4: Suppose there are three piles of stones numbered 2, 5, and 6, A and B play this game. A starts.

| Number of stones on the first piles | Number of stones on the second piles | Number of stones on the third piles | Steps of players |
|---|---|---|---|
| 2 | 5 | 6 | The game starts. |
| 2 | 4 | 6 | A starts and takes 1 stone from the second pile. |
| 1 | 4 | 6 | B is next and takes 1 stone from the first pile. |
| 1 | 4 | 5 | A takes 1 stone from the third pile. |
| 1 | 4 | 1 | B takes 4 stones from the third pile. |
| 1 | 0 | 1 | A takes all 4 stones from the second pile. |
| 0 | 0 | 1 | B takes 1 stone from the first pile. |
| 0 | 0 | 0 | A takes the last stone from |

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the third pile, and A wins.

A wins the game again, but the strategy used is not as obvious as in the previous example, and the way A proceeds seems different from that in the game with two piles. The way to clarify A's strategy is to divide each number of stones into different powers of 2 and focus on A's moves again.

Table 5: Suppose there are three piles of stones numbered 2, 5, and 6, A and B play this game. A starts.

| Number of stones on the first piles | Number of stones on the second piles | Number of stones on the third piles | Steps of players |
|---|---|---|---|
| 2 | 5 | 6 | The game starts. |
| 2 | 4 | 6 | A starts and takes 1 stone from the second pile. |
| 1 | 4 | 6 | B is next and takes 1 stone from the first pile. |
| 1 | 4 | 5 | A takes 1 stone from the third pile. |
| 1 | 4 | 1 | B takes 4 stones from the third pile. |
| 1 | 0 | 1 | A takes all 4 stones from the second pile. |
| 0 | 0 | 1 | B takes 1 stone from the first pile. |
| 0 | 0 | 0 | A takes the last stone from the third pile, and A wins. |

The strategy used by A is now clearer. Although the number of stones in each stack appears to be different, by breaking them, each number of different powers of 2 is even, so the number of stones in the piles is the same in a hidden way. In example three, after the first time A takes stones, there is a pair of 2 and 4 that allows A to follow B's steps and win.

However, there are some cases that cannot be solved by subtraction alone. Here is example 4, whose pile contains 2, 3 and 6 stones. Decomposing them into different powers of two yields these equations.

$$2 = 2 \qquad 3 = 1 + 2 \qquad 6 = 2 + 4$$

There are three 2s, one 1, and one 4. None of these numbers has an even number. What should A do then? First, A should start before B so that the piles are "equal". The next step is to choose the piles, which is key to this situation. The crux of the matter is that stones can not only be subtracted from the piles, but also added to them face down. For example, if we want to subtract 4 from the third pile and add 1 to make all the piles "equal", A must subtract 3 stones from the third pile, because

$$-4 + 1 = -3$$

In this way, players can add stones to the stacks. The reason A has to take stones from the third pile is that it is impossible for the largest number 4 in the third pile to be added to any other pile, because all the other two piles contain less than 4 stones, and adding 4 stones to those piles would exceed the original number of stones in those piles. Example: A wants to add 4 stones to the second pile, while subtracting all stones from the first pile. Since

$$-3 + 4 = 1$$

A must get an additional stone for the second pile, thus obeying the rule. On the other hand, it seems possible to add 1 stone to either of the other two stacks, and by subtracting any 2 stones in any stack, the remaining 2 become a pair. 2 is out of the question because there are 2 in all piles. Therefore, 2 can be paired no matter which pile A takes a stone from. If A wants to work with the

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second pile, subtracting 1 and add 4 is also necessary, because the left 1 and the 4 are not in the same pile and cannot be taken away in one move. Then A gets into the unsolvable situation that 3 additional stones are needed for the second pile, because

$$-1 + 4 = 3$$

So A has to take 4 stones from the third pile to get rid of 4. A must also subtract 2 stones from the third pile to make 2 pairs. Finally, 1 stone must be added to make it a pair if it is decided that 4 should be subtracted. Consequently, A should choose the first move and subtract 5 stones from the third pile.

$$-4-2+1=-5$$

Table 5 shows how this method works.

Table 5: Suppose there are three piles of stones numbered 2, 3 and 6 and A and B are playing this game. A starts.

| Number of stones on the first piles | Number of stones on the second piles | Number of stones on the third piles | Steps of players |
|---|---|---|---|
| 2 | 3=1+2 | 6=2+4 | The game starts. |
| 2 | 3=1+2 | 1 | A starts and takes 5 stones from the third pile. |
| 2 | 0 | 1 | B is next and takes 3 stones from the second pile. |
| 1 | 0 | 1 | A takes 1 stone from the first pile. |
| 1 | 0 | 0 | B takes 1 stone from the third pile. |
| 0 | 0 | 0 | A takes the last stone from the first pile, and A wins. |

Table 6 shows that adding is possible even if The Game of Nim is a subtraction game. So the next time there is an even more difficult task with the numbers 1, 2 and 8, just solve it with addition.

$$-8 + 1 + 2 = -5$$

Taking 5 stones from the third pile and following the steps of another player would be the winning strategy for the three-pile game.

After discussing the Nim game with two piles and three piles, let us test the winning strategy used in the three-pile game in the four-stack game to find a common winning strategy for the Nim game. Here is Table 6, which offers four piles of stones numbered 2, 3, 7, 9. If A follows the winning strategy in the three-pile game, the game will proceed as in Example 5.

Table 6: Assuming there are four piles of stones with numbers 2, 3, 7, and 9, A and B are playing this game. A goes first.

| Number of stones on the first piles | Number of stones on the second piles | Number of stones on the third piles | Number of stones on the fourth piles | Steps of the players |
|---|---|---|---|---|
| 2 | 3=1+2 | 7=1+2+4 | 9=1+8 | The game starts. |
| 2 | 3=1+2 | 7=1+2+4 | 6=2+4 (-1-8+2+4=-3) | A goes first, taking 3 stones from the third pile. |
| 2 | 0 | 7=1+2+4 | 6=2+4 | B goes next, taking 3 stones from the second pile. |
| 2 | 0 | 4 | 6=2+4 | A takes 3 stones from the third pile. |

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| | | | | |
|---|---|---|---|---|
| 2 | 0 | 4 | 2 | B takes 4 stones from the fourth pile. |
| 2 | 0 | 0 | 2 | A takes 4 stones from the third pile. |
| 0 | 0 | 0 | 2 | B takes 2 stones from the first pile. |
| 0 | 0 | 0 | 0 | A takes 2 stones from the fourth pile, and A wins. |

According to Example 5, the strategy of the three-pile game can also be used in the four-pile game. A can still make all piles "equal" with this strategy. From this, it can be deduced that the winning strategy for The Game of Nim is to use addition and subtraction according to the result of the different powers of 2 fractioned by the number of stones of each pile to make the numbers of each fractional number even and follow the steps of the other player to win the game.

To explain the principle behind this strategy, a term called "Nim-sum" should be introduced. Nim-sum is a type of calculation with the sign "$\oplus$". There are some steps to calculate nim-sums. For example, for $3\oplus 6$=?, here are the steps to solve it

Step 1: Decompose each number into different powers of 2.

$$3 = 1 + 2 \quad \oplus \quad 6 = 2 + 4$$

Step 2: Cross our the paired fractional numbers.

$$(1 + \cancel{2}) \quad \oplus \quad (\cancel{2} + 4)$$

Step 3: Add all the left numbers.

$$1 + 4 = 5$$

So the result of $3\oplus 6$ =? is 5.

With the knowledge of the nim-sum, the process of making piles "equal" in the game can now be translated as "the nim-sum is equal to zero." Consequently, the winning strategy for The Game of Nim is to make the nim-sum of the number of stones in the piles equal to zero every time.

## 3. The Subtraction Game

The Subtraction Game is a similar version to the original game in terms of rules. What does not change is that this game also requires two participants and the one who gets the last stone wins. The differences, however, are that there is only a single pile of stones and that there is a limit to the number of stones each player can take per turn. Here is an example.

Table 7. There are 10 stones and each player can take no more than 3 stones per turn. A starts.

| The number of stones left | Steps of players |
|---|---|
| 10 | The game starts. |
| 7 | A starts and takes 3 stones away. |
| 4 | B starts and takes 3 stones away. |
| 2 | A takes 2 stones. |
| 0 | B takes 2 stones and B wins. |

How can A win? If there are 3 or less stones left when it is A's turn, A can win the game. If there are exactly 4 stones left when it is B's turn, A can win the game...

Reasoning Step 1:

When it is A's turn, there should be 1 to 3 stones left for A to win the game.

Reasoning Step 2:

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When it is B's turn, there should be 4 stones left so that B cannot take all the stones away. But no matter how many stones B takes, there will be 1 to 3 stones left for A to win.

Reasoning Step 3:

On A's penultimate turn, there should be 5 to 7 stones left so that 4 stones remain for B to win.

Reasoning Step 4:

When B's turn is second to last, there should be 8 stones, so that no matter how many stones B takes, there will be 5 to 7 stones left for A.

Reasoning Step 5:

There should be 9 to 10 stones at the start so that A leaves 8 stones for B.

After reasoning, how will the game go if you follow the steps? Here comes the example.

Table 8: There are 10 stones and each player can take away no more than 3 stones per turn. A starts.

| The number of stones left | Steps of players |
|---|---|
| 10 | The game starts. |
| 8 | A starts and takes 2 stones away. |
| 7 | B starts and takes 1 stones away. |
| 4 | A takes 3 stones. |
| 2 | B takes 2 stones. |
| 0 | A takes 2 stones and A wins. |

Example 7 shows that the thought process is correct. Similarly, we can develop the winning strategy for The Subtraction, where the number of stones is M and the largest number of stones that the player can take away is N.

Reasoning Step 1:

In A's last move, there should be 1 to N stones for A to win the game.

Reasoning Step 2:

There should be (N+1) stones when it is B's last turn, so B cannot take all the stones away, but no matter how many stones B takes, there will be 1 to N stones left for A to win.

Reasoning Step 3:

There should be (N+2) to (2N+1) stones when A makes its penultimate move so that (N+1) stones remain for B.

Reasoning Step 4:

There should be (2N+2) stones when B makes its penultimate move, so that no matter how many stones B takes, (N+2) to (2N+1) stones remain for A.

Reasoning Step 5:

There should be (2N+3) to (3N+2) stones when A makes its third to last move so that (2N+2) stones are left for B.

Reasoning Step 6:

There should be (3N+3) stones when B makes its third last move, so that no matter how many stones B takes, (2N+3) to (3N+2) stones remain for A.

During the reasoning process, it can be seen that A needs a multiple of (N+1) stones to win the game. Therefore, A should go after B if M is a multiple of (N+1), and go first so that the number of stones is a multiple of (N+1) if M is not. Also, A must make sure that the sum of the number of a pair of A and B's stpes is (N+1).

## 4. Application

Having learned the principles and strategies of the original version of the Game of Nim and the Subtraction Game, where to apply them need to be considered. Is there a game that has been played many times in life that covers the principles of these two games? At this point, the game "The 21 Game" should be discussed.

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Any number of competitors (at least 2) can join the 21 Game. The players have to take turns saying a number. The first player starts with the number 1, and the others can add 1, 2, or 3 in each round until someone who has to say a number greater than or equal to 21 loses the game. In a game with 2 players, this can be modeled as a Subtraction Game with the gameplay reversed. Adding numbers is the same as subtracting numbers and the winning goal can be changed to say 20, so the winning strategy should be to go next and always end the round by saying a multiple of four. Example 8 shows how the strategy works.

Table 9. The 21 Game sample

| The number said by the players | Steps of players |
|---|---|
| 1 | The game starts, A starts with 1. |
| 1+3=4 | B is next and adds 3. |
| 4+2=6 | A adds 2. |
| 6+2=8 | B adds 2. |
| 8+3=11 | A adds 3 |
| 11+1=12 | B adds 1 |
| 12+3=15 | A adds 3 |
| 15+1=16 | B adds 1 |
| 16+2=18 | A adds 2 |
| 18+2=20 | B adds 2 |
| 20+3=23 | A adds 3 and exceeds 21, so A loses. |

Table 9 shows how the winning strategy of the Subtraction Game works in The 21 Game when only 2 players are involved. However, the strategy must be changed when more than 2 players are involved.

## 5. Conclusion

This essay is about the winning strategies of The Game of Nim and one of its different versions, the Subtraction Game. The winning strategy of the Game of Nim is to bring the nim-sum of the stones in the piles after your turn to zero by addition and subtraction, and the winning strategy for the Subtraction Game is to bring the number of stones after your turn to a multiple of (N+1) until you have only 1 to N stones left for another player, when the number of stones you can take away each time is limited to N. This is the winning strategy of The Game of Nim. Moreover, the strategy of the theoretical model can be applied to real games of chance like The 21 Game.

The Game of Nim is just a simple case study in game theory, where there are many other basic ideas, such as information asymmetry and adverse selection. Similarly, these basic ideas can be widely applied in real life. In the future, we will also be able to analyze more cases in game theory, so that more theoretical models from game theory can be applied to real economic and social life.

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