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The Application of Auxiliary Problem in Solving Trigonometric Function Problems In High School

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Abstract. In order to improve the efficiency of high school students in solving problems and provide them with problem-solving ideas, this article uses the method of finding auxiliary problem proposed by Polya, and the means of solving complex problems is studied by taking trigonometric function of high school as an example. Through analysis, it clarifies the applicable types of questions, advantages, and application difficulties of each method, and provides some helpful suggestions for solvers..

Keywords: auxiliary problem; high school; trigonometric function.

1. Introduction

Mathematical problem-solving ability may reflect students' understanding of mathematical knowledge, in solving the problem, auxiliary problem plays an important role (Kuzu, 2022; Muzaini, Rahayuningsih, Ikram et al., 2023). Kuzu (2023) contends that teachers currently have a variety of problems in the use of auxiliary problem in the teaching , which leads to students' lack of understanding of knowledge. This essay will therefore provide some case references for teachers to use auxiliary problem reasonably in teaching.

2. Related Words

2.1 The concept of auxiliary problem

The concept of auxiliary problem was put forward by Polya in 2011, he asserted that the importance of auxiliary problem was not itself, but in the fact that solvers could solve another complex problem by thinking about it, which was the problem they originally wanted to solve. Actually, auxiliary problem is a means for solvers to achieve their goal, and the original problem is what they want to achieve. When the problem solvers are confused about the original problem and cannot find the answer directly, they need to use some tools to help them, at this time, finding a auxiliary problem to solve is a better way.

2.2 The significance of auxiliary problem

The auxiliary problem is enlightening. For problem that is difficult to solve, auxiliary problem actually provide the solvers with familiar method. Ausubel (1968) argued that meaningful learning materials could meet the requirements of establishing substantial and impersonal connections with the relevant knowledge in the cognitive structure. The auxiliary problem is to use the method that conforms to the original cognition of the solvers, establish the connection between the auxiliary problem and the original problem, and solve the unfamiliar problem with the familiar one, which aims to build a bridge between the old and new knowledge, and help the problem solvers to deeply understand the original problem.

3. Problem Formulation

Functions are the core knowledge of high school mathematics and difficult concepts for students to understand (National Council of Teachers of Mathematics, 1989). Therefore, there are many researches on teaching and learning related to function problem solving. Introducing auxiliary function not only facilitates the problem solving , but also exercises students' thinking skills and develops their intelligence (Jiang & Li, 2010). However, the ability of high school students to draw inferences and solve problems from one instance to others still needs to be strengthened, which is based on deep thinking about the Function concept (Ministry of Education in China, 2017).

4. Analysis of Examples

In Polya's book "How to Solve Problems", he talked about the method of finding auxiliary problem: first, observe the unknown quantity and find some useful hints. Next, combine the changed problem with the hints to obtain useful auxiliary problems. There are many ways to change the problem: decomposition and recombination, introducing auxiliary elements, generalization, specialization, analogy, equivalent problems, equivalent auxiliary problem chains, etc. Therefore, taking trigonometric functions in high school mathematics as an example, this research shows how to find auxiliary problem and how to use them to solve problems.

4.1 Decomposition and recombination

The first step in solving a complex mathematical problem is to understand the problem in depth, breaking it down into three small questions: What are the unknowns? What are the known data? What are the conditions? The second step is to study each piece of data and separate the different parts of the condition, and study each part individually. The third step is to recombinate the elements using new methods to form a new, easier-to-approach problem.

Example 1: The opposite sides of the interior angle A,B,C of $\vee ABC$ are a,b,c. Given that $\sin\left(A+C\right)=8\sin^2\frac{B}{2}$.

Find the value of $\cos B$.

(1) If $a+c=6$ and the area of $\vee ABC$ is 2, what is the value of b?

Analysis and solution: Omit the answer to the first question and look for the answer to the second question.

① Understand the question and break it down into three small questions.

Unknown: b;

Known: The opposite sides of the interior angle A,B,C of $\vee ABC$ are a,b,c;

Conditions: $a+c=6$, the area of $\vee ABC$ is 2.

② Analyze and recombinate the problem.

In order to solve b, thinking of using the Cosine Theorem $b^2=a^2+c^2-2ac\cos B$; and knowing $a+c=6$, solver could use the Perfect Square Formula, at this time, a^2+c^2 could be expressed in terms of ac, which is $a^2+c^2=36-2ac$, and then plugging it into $b^2=a^2+c^2-2ac\cos B$, and the solver get a new formula $b^2=36-2ac-2ac\cos B$.

Besides, knowing the value of $\cos B$,If the solver could solve the value of ac, the problem can be solved. At this point, the problem changed from finding the value of b to finding the value of ac.

In fact, the value of ac is solved based on two conditions. first, the area of $\vee ABC$ is 2. Second, Find the value of $\sin B$ from the value of $\cos B$ in the first question.

Therefore, reorganizing the problem after the Analysis:

Unknown: ac;

Known: The opposite sides of the interior angle A,B,C of $\vee ABC$ are a,b,c, and $\cos B=\frac{15}{17}$;

Conditions: the area of $\vee ABC$ is 2.

The auxiliary problem of the original question is the opposite sides of the interior angle A,B,C of ∇ABC are a,b,c , $\cos B=\frac{15}{17}$ and the area of ∇ABC is 2. what is the value of ac ?

There are many advantages to decomposition and recombination. Firstly, allowing the solvers to deepen their understanding of the known quantities, conditions, and unknowns, and make it easier for the solver to find connections among the three. Secondly, it allows the solvers to be more specific about all the unknowns that need to be solved. Thirdly, it helps solvers understand confusingly known quantities. Through thinking about the above problem solving, it could be found that this type of problem usually needs to understand, analyze and reorganize the problem information to find a solution. A known quantity often contains many hidden information, and when solving a problem, it is necessary to first solve the hidden information as an auxiliary problem, and then solve the final problem after obtaining the hidden information. The above topic is an example, and the solution is shown in Figure 2-1:

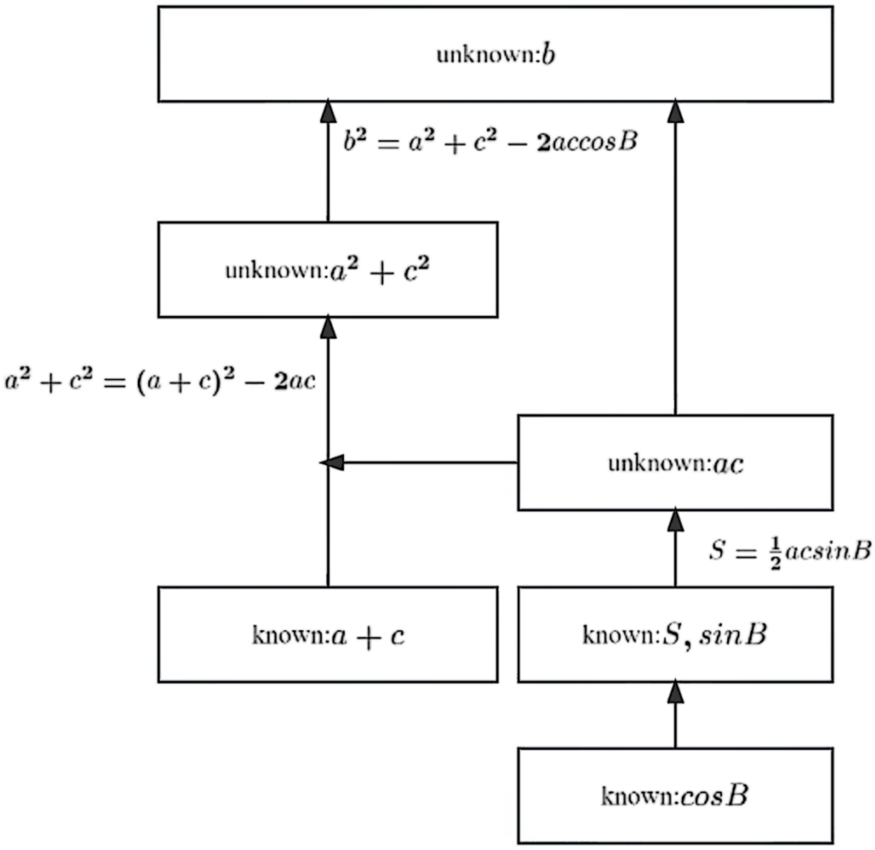

Figure 2-1

4.2 Introducing auxiliary elements

Polya (2011) states that auxiliary elements as elements introduced to facilitate the solution. Different types of problems require different auxiliary elements. New lines drawn in geometry problems are auxiliary lines, auxiliary elements added in algebra problems are auxiliary unknowns, and theorems introduced in proof problems to help prove the theorem are auxiliary theorems. The basis of mastering auxiliary elements is to have a certain accumulation of knowledge, in order to think of what auxiliary elements to use to answer.

Example 2: Given the vector $\boldsymbol{a}=(\sin 2x,\cos 2x)$, vector $\boldsymbol{b}=\left(\cos\frac{2\pi}{3},-\sin\frac{2\pi}{3}\right)$, and $f(x)=\boldsymbol{a}\cdot\boldsymbol{b}-2\sqrt{3}\cdot\sin^2\left(x-\frac{\pi}{3}\right)+\sqrt{3}$.

(1) Find the range of $f(x)$ and the center of symmetry of $f(x)$ image.

(2) If the equation $f(2x)-a=0(a\in\mathbf{R})$ has two different solutions x_1, x_2 in the interval $\left[0,\frac{\pi}{4}\right]$, what is the value of $\sin(x_1+x_2)$?

Analysis and solution: Omit the answer to the first question and look for the auxiliary problem to the second question.

Step 1: Suppose that the solver has already found the expression for $f(x)$, so $f(2x)=2\sin\left(4x-\frac{\pi}{3}\right)$, based on the range of values of x, can find the range of values of $4x-\frac{\pi}{3}$.

Step 2: Since $4x-\frac{\pi}{3}$ is an undivisible whole, try to call it as t, at this time $t=4x-\frac{\pi}{3}$, and $f(t)=2\sin t$.

Step 3: Change the original question into a new auxiliary problem: Given the equation $f(t)=a(a\in\mathbf{R})$ has two different solutions t_1, t_2 in the interval $\left[-\frac{\pi}{3},\frac{2\pi}{3}\right]$, find the value of t_1+t_2. Now, t_1 and t_2 could be represented in terms of x_1 and x_2, and from the graph properties of $y=2\sin t$, knowing $t_1+t_2=\pi$, which gives solver the value of x_1+x_2, and finally the value of $\sin(x_1+x_2)$.

The example above shows that the most important thing to do in introducing auxiliary elements is to observe carefully and combine new things in the problem with familiar knowledge. There are many benefits to this approach. First of all, by changing the idea, the original complex formula could be simplified to facilitate calculation. Second, master a common thinking mode and method for solving mathematical problems, that is, transform unfamiliar knowledge into familiar knowledge. Third, help solvers make connections between old and new knowledge.

4.3 Equivalent auxiliary problem

Equivalence problem means that if each solution involves the other's solution, then the two problems are equivalent (ibid). The process of changing from the original problem to the auxiliary problem is called reversible reduction, or bidirectional reduction, or equivalent reduction.

Example 3: Given that the graph of the function $f(x)=a\cos x+\sqrt{3}\sin x$ is symmetric about the center of the point $\left(\frac{\pi}{6},0\right)$, shorten the horizontal coordinate of each point on the curve $y=f(x)$ to the original $\frac{1}{2}$, the ordinate is unchanged, and shift the curve to the left by a unit length of $\frac{\pi}{6}$ to get the curve $y=g(x)$, where $h(x)=g(\omega x)(\omega>0)$, if $h(x)$ has exactly 7 zeros on $[0,2\pi]$, Find the range of ω.

Analysis and solution: Knowing that $h(x)=g(\omega x)=2\sin\left(2\omega x+\frac{\pi}{6}\right)$, and $x\in[0,2\pi]$ the solver could add new auxiliary element, support $t=2\omega x+\frac{\pi}{6}(\omega>0)$, could obtain a range of values for t is $\left[\frac{\pi}{6},4\pi\omega+\frac{\pi}{6}\right]$, and $h(x)$ becomes $y=2\sin t$. In this case, $h(x)$ has exactly 7 zeros on $[0,2\pi]$ is equivalent to $y=2\sin t$ has exactly 7 zeros on $t\in\left[\frac{\pi}{6},4\pi\omega+\frac{\pi}{6}\right]$. Find the range of $h(x)$.

In general, for a complex trigonometric function, it is not easy to find the zero, because its image is not familiar to the solver, and it is relatively difficult to draw a picture. Therefore, it is necessary for solvers to find equivalent auxiliary problem.

4.4 Specialization

Equivalence problem is a kind of bidirectional reduction. However, specialization belongs to unidirectional reduction. Polya (2011) explains that there were two unsolved problems A and B. If solvers could solve A, then they could get a complete solution for B; But the reverse is not true, if individuals could solve B, they may be able to get some information about A, but cannot get the complete solution of A from the solution of B. The solvers could think of A as the one with the higher expectation and B as the one with the lower expectation. If solvers move from the original problem to a secondary problem with higher expectation or a secondary problem with lower expectation, this step is called unidirectional reduction. In order to facilitate understanding, the article gives a few examples.

Example 4: Given that the length of three edges of a cuboid led by a vertex is a,b,c, find the diagonal length of this cuboid.

Analysis and solution: The diagonal formula for a cuboid is $\sqrt{a^2+b^2+c^2}$, which, when $c=0$, is the diagonal of a rectangle $\sqrt{a^2+b^2}$.

In this problem, the cuboid diagonal line has the higher expectation, while the rectangular diagonal line has the lower expectation, and the problem with a larger expectation needs to be solved with the help of the problem with a lower expectation, which is the specialization to find auxiliary problem.

Example 5: Prove the law of cosine.

The answer: CD is the height of the AB side of the $\triangledown ABC$. According to Pythagorean theorem:

$$
\begin{aligned}
BC^2 &= CD^2+BD^2\\
&=(AC\cdot\sin A)^2+(AB-AC\cdot\cos A)^2\\
&=AC^2\sin^2 A+AB^2+AC^2\cdot\cos^2 A-2AB\cdot AC\cdot\cos A\\
&=AB^2+AC^2-2AB\cdot AC\cdot\cos A
\end{aligned}
$$

Analysis and solution: There are many methods to prove the law of Cosine, but the Pythagorean theorem is used to show how to solve problems with lower expectations by using problems with higher expectations. In fact, in the proof of the cosine theorem, when $\angle A=90°$, it becomes Pythagorean theorem.

4.5 Generalization

Generalization is also unidirectional reduction. Unlike specialization, generalization is unidirectional reduction of a problem with a larger expectation (ibid). In fact, problems with higher expectations are often easier to solve, which is the Creator paradox. For example, the solution to a problem is more difficult than multiple problems, more comprehensive theorems are easier to prove, and more general problems are easier to solve. That is why we need to use generalization to solve the problem.

Example 6: The opposite sides of the interior angle A,B,C of acute triangle A,B,C are a,b,c. Given that $B=2A$. Find the range of $\frac{b+c}{a}$.

Analysis and solution: This problem requires the solvers to work out the range of edges, but the information in the problem is all about angles. So they could take a wild guess and convert $\frac{b+c}{a}$ into an angular formula by the sine theorem. After that, gradually reduce the number of angles in the equation. According to $C=\pi-\left(A+B\right)$ and $B=2A$, it is assumed that the final expression must contain only Angle A or Angle B. Then according to the condition of the acute triangle, solvers could know the range of $\sin A$ and $\sin B$, and finally plug in the algebra of the Angle in the first step, this problem could be solved completely. Therefore, this problem could be generalized as given the range of $\sin A$, find the range of the expression containing $\sin A$.

As could be seen from the above example, the search for generalized auxiliary problems usually starts with the conclusion, looking for the general law. Once the generalized auxiliary problem is found, the problem solver has found the root of this type of problem, called the Original Problem. In fact, many problems are based on the original problem to do various transformations.

5. Conclusion and Suggestion

By analyzing the above examples, it could be concluded that the problem solvers should understand the problem at first, and know the unknown quantities, data and conditions. Secondly, find the connection between known quantities and unknown quantities, and if no connection could be found, establish the connection with the auxiliary question. This research lists 5 methods to find auxiliary problem, and Table 3.1 shows these methods and the applicable problem types, advantages, and difficulty in application of these methods.

Table 3.1

Method	Applicable problem types	Advantages	Difficulty in application
Decomposition and Recombination	Applies when change the condition or conclusion, or both, so that the new condition makes it easier to derive the new conclusion.	Conduce to clarify the relationship between known data, conditions, and unknowns.	Identify the unknowns and conditions and find the necessary relationships between them.
Introducing Auxiliary Elements	Applicable when several parts of the problem are the same whole or could be transformed into familiar content by introducing auxiliary elements.	Simplify the problem, use known knowledge to solve unknown knowledge, and form knowledge transfer.	Auxiliary elements are harder to find.

Equivalent Auxiliary Problem	Applicable when known condition could change to other condition and the new one gradually approach the conclusion.	Helps discover hidden information in known conditions.	Difficult to find the transformation method of equivalent transformation with known conditions.
Specialization	Applicable when problems with higher expectations could be solved with the help of problems with low expectations.	Problems with higher expectations could be answered by problems with lower expectations.	Difficult to find problems with higher expectations that correspond to problems with lower expectations.
Generalization	Applicable when problems with lower expectations could transition to problems with higher expectations.	Finding the original problem, and the underlying theorem for the given problem.	Original topics are hard to find and require a lot of basic knowledge.

In summary, almost every method of finding auxiliary problem requires a lot of basic knowledge. Jiang and Li (2010) put forward the theory of The Nearest Development Zone, which shows that the auxiliary problem is a bridge between the old and the new knowledge, through which the solvers could inspire thinking and improve the efficiency of solving the problem. In addition to enriching the original knowledge, the solvers need to deeply understand the connotation of each method. After fully understanding the above methods, the problem solvers may also innovate and seek new ways to find auxiliary problem, and then internalize their own unique problem-solving style.

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