

Why Arithmetic's Incompleteness Might Suggest That Mathematics Is Discovered

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Abstract. Is mathematics invented or discovered? This question has perturbed philosophers and mathematicians alike for centuries. In this manuscript, we propose a novel argument in favor that fundamental arithmetic is discovered rather than invented using Kurt Gödel's Incompleteness Theorem because essentially incompleteness seems to be a property unique to arithmetic and counter-intuitive to invented systems. We first summarize Gödel's argument, then argue that arithmetic's essential incompleteness shouldn't be a property in purely invented systems without it, giving an example on how to resolve incomplete systems in those systems and argue that arithmetic is to some extent discovered because of this difference in property. We also conjecture that any invented system that could be proven to be essentially incomplete with a similar logic to Gödel's method can express arithmetical relationships in some form. Finally, we account for some objections to discovery.

Keywords: philosophy of math, Godel, Godel numbering, theories of arithmetic, axiomatizable theories, incompleteness, essential incompleteness, is math discovered or invented.

1. Introduction

I first encountered the question "Is math discovered or invented?" in a philosophy and math club joint meeting earlier this year. The implied clash between the two sides is whether mathematical truths depend on human intelligence. There is no question that the language we use to describe arithmetic is invented. However, through this language, are we creating or merely interpreting? Before the meeting, because of my skepticism toward universal objectivity, I had been pre-disposed to favor that math was invented instead of discovered. However, that opinion was swayed by the end of the meeting.

Philosophers and mathematicians throughout history have attempted to answer variations of this question. The ancient Greeks mostly agreed that it is discovered, most notably the Pythagoreans who believed everything is numbers and Plato who believed that mathematics could bridge the real world and the abstract world of forms (Barker 1994). Eugene Wigner, with his famous phrase, "the unreasonable effectiveness of mathematics," argued that the sheer

wide and sometimes unexpected applicability of mathematical concepts in other disciplines and the real world (Wigner 1995). Even babies have an innate number sense (Dehaene 2011). Since we are arguing on the subjective influence of humans, we could also imagine if an alien civilization could have a different mathematical system. It seems almost impossible that they won't have some way or another of expressing natural numbers.

On the other hand, people who believe that math is invented could argue that the specific way we "do" math is not unique. Alien civilizations might have some notation of numbers, which might not be separate and abstract concepts as our numbers are but rather depend on the objects they describe. It is just like how a baby might understand that three M&Ms is better than two M&Ms but might not find a shared property between three M&Ms and three buildings (Guerrini 2023). They definitely would not share our mathematical language or might not even have fields corresponding to our calculus or trigonometry. In addition, if math is discovered and thus universal, then why does the way we build our fundamental mathematical system change over the ages? Our current axioms and logical theory defining math differs greatly from how our ancestors understood math two thousand years ago. Wouldn't that suggest that math is subjective and subject to our change? There are also mathematical concepts that depend on humans, such as the Erdos Number (Goffman 1969), which even the most adamant of the "discoverists" would have to concede is more invented than not.

Both sides of the argument were fascinating, but I was leaning more towards invention. However, then Gödel’s Incompleteness Theorem was brought up. In the 1920s, Kurt Gödel stunned the mathematical world with his incompleteness theorems, proving that math is condemned to be incomplete, that there will always be unprovable but true sentences in any consistent logical theory capable of arithmetic, ruining the dreams of Bertrand Russell and Alfred Whitehead of constructing a perfect mathematical system.

The incompleteness theorem proved that our understanding of math could only be hopelessly imperfect. It was and will always be a bleak conclusion and it felt...wrong. However, out of that bleakness and intense counter-intuition intuition sparked. Wouldn’t that hopeless imperfection and the Book of Math’s stubborn insistence on never being read completely suggest that it is above our understanding? And if it is above our understanding, wouldn’t that imply that we cannot invent it? It isn’t that arithmetic isn’t consistent, but we just cannot demonstrate its consistency. The fact that arithmetic is a sufficient condition for incompleteness further convinced me of its uniqueness.

Later, with more research, J.R. Lucas and Roger Penrose had made a similar argument by using the incompleteness of math to argue that artificial intelligence could never think like a human. Every machine we have could be replaced by a Turing machine. The Turing Machine, though not a logical theory by itself, exhibits a property similar to incompleteness in the unsolvability of the halting problem. (Burkholder 1987) Computers and by extension artificial intelligence are incomplete. Lucas and Penrose argues that our minds are not machines because we don’t have this fundamentally “unprovable sentence” in our minds (Penrose 2016). This argument has a lot of potential problems because of how little we know of the nature of our minds and consciousness. However, our logic is similar to his and we shall argue that the incompleteness of math suggests that it is partly discovered rather than purely invented because purely invented systems would lack the incomplete property that arithmetic systems possess.

2. Proof of Gödel’s Incompleteness Theorem

Here we shall give a short summary of Gödel’s process that will tie into my argument on the “discoveredness” of mathematics.

2.1 Gödel Numbering

Gödel’s proof relies on forming an injection from the set of theorems in an arithmetic system onto the set of all natural numbers. Every arithmetical sentence could be expressed with a finite sequence of logical symbols from a limited set. He then assigned each symbol a representative number, for example in Table 1 (Davey 2023)

Table 1: Arithmetical Symbols to Gödel Number Representative

symbol	meaning	number
0	0	0
·	successor (e.g.0' = 1)	1
+	plus	2
.	multiply	3
=	equal	4
∨	and	5
∧	or	6
—	not	7
∃	there exists	8
∀	for all	9
(left parentheses	10
)	right parentheses	11
v_i	i^{th} variable	$11 + i$

This would effectively map every arithmetical sentence to an ordered set of numbers. For example the claim $1 + 1 = 2$ or $0' + 0' = 0''$ would be represented as $(0, 1, 2, 0, 1, 4, 0, 1, 1)$. We could then derive the theorem's so-called "Gödel number" by raising the i th prime to the power of the representative number of the i th symbol in the theorem. In the example $1 + 1 = 2$, its Gödel number would be $2^0 \cdot 3^1 \cdot 5^2 \cdot 7^0 \cdot 11^1 \cdot 13^4 \cdot 17^0 \cdot 19^1 \cdot 23^1 = 10296954525$. Though it seems way too aggrandizing to express an expression as simple as $1 + 1 = 2$ with an 11-digit integer, this means that every single expression using these 11 symbols and finite variables could be encoded into a single number.

2.2 Diagonalization Lemma

Through assigning a unique number for statements and even proofs, Gödel was able to prove the existence of logical statements that are equivalent to statements about their own Gödel numbers known as the Fixed Point Theorem, or the Diagonalization Lemma (Davey 2023):

Theorem 1 (Diagonalization Lemma). For any unary formula $\psi(x)$ definable in the language of arithmetic, there exists a sentence S in the language of arithmetic such that: $S \leftrightarrow \psi(\ulcorner S \urcorner)$ where $\ulcorner S \urcorner$ denotes the Gödel number of sentence S .

This is crucial in his proof as it allows the creation of pseudo self-referential statements about the statement's own Gödel number. It is because logical sentences in arithmetic can form an injection onto the natural numbers and be processed in its own system that this is possible.

2.3 Definability of Unprovability

With the Diagonalization Lemma, we only need to prove that the property of unprovability is definable in arithmetic to substitute it for $\psi(x)$ and get the Gödel sentence. Gödel was in fact not only able to prove that unprovability was definable in arithmetic, but in all logical theories where the axioms are Turing enumerable (Gödel 1931):

Theorem 2 (Definability of Unprovability). For every axiomatizable theory T , the relation $\text{Prov}_T(z)$ is definable.

The reasoning behind this theorem is simple. We only need to mechanically list out all the possible proofs in theory T and look if there exists a proof for which z is the conclusion. With this in hand, we can finally get the Gödel sentence.

2.4 Gödel's First Incompleteness Theorem

2.4.1 Definition of Incompleteness

System T is complete, or negation complete iff for every valid logical sentence S in system T , $(T \vdash S) \vee (T \vdash \neg S)$.

2.4.2 The Gödel Sentence

The Gödel sentence is as follows (Gödel 1931):

$$G \leftrightarrow \neg \text{Prov}_T(\ulcorner G \urcorner) \tag{1}$$

In other words, G is equivalent to its own unprovability. Gödel assumed that the Gödel sentence was true because its falsehood would lead to a contradiction. If G is not provable in system T , it would be provable and imply its truth in T . But we have established its falsehood, and thus there is a contradiction. G could only be true and unprovable. Math is thus incomplete.

In fact, Gödel proved that in any formal system F that is capable of performing a certain amount of arithmetic there exists true statements that could not be proven. He derived a mechanical method to construct a true but unprovable sentence in any arithmetical theory with Turing enumerable axioms. In other words, arithmetic capability is a sufficient condition for incompleteness. Arithmetic is essentially incomplete.

2.5 Significance of Gödel's Incompleteness Theorems

In essence, Gödel's proof utilizes the inherent order and universality of the natural numbers to create a mechanical method for quasi self-reference. If system I is not capable of arithmetic, it wouldn't have the capability of constructing semi-self-referential sentences such as the Fixed Point Theorem. In other

words, we won't have a mechanical method to construct a true but unprovable sentence for purely invented systems that will exist no matter how many axioms we add to the system. This unpergeable persistence of incompleteness, known as essential incompleteness, is fundamental to the nature of arithmetic itself.

3. Why Arithmetic's Incompleteness Might Imply that Math is Discovered

3.1 The Complete Baby Arithmetic

One valid question is whether all arithmetical systems are incomplete. The answer is no. Consider the following quantifier and connective free system Baby Arithmetic based on the following schemas where ζ and ξ can be systematically replaced with numerals to derive the set of all axioms of Baby Arithmetic:

Schema 1. $0 \neq S\zeta$

Schema 2. $S\zeta = S\xi \rightarrow \zeta = \xi$

Schema 3. $0 + \zeta = \zeta$

Schema 4. $\zeta + S\xi = S(\zeta + \xi)$

Schema 5. $0 \times \zeta = 0$

Schema 6. $\zeta \times S\xi = \zeta \times \xi + \zeta$

where Sx indicates the successor of x . Note that these 6 schemas are not by themselves the axioms of Baby Arithmetic, but the set of all numerical replacements with the schemas. Baby Arithmetic is negation complete. It can prove the equality of every equal equation and prove the inequality of every unequal equation in its language because its language is purely based on logical connectives connecting finite truth variables. With the rules of inference and truth tables, like classic propositional logic, we can evaluate the truth value of every claim mechanically.

However, Baby Arithmetic cannot express numerical generalizations. It can only determine the truth value of instance equations. Its stronger version, known as Robinson Arithmetic, replaces all of Baby Arithmetic's schemas with numerical generalizations (e.g. $(Ax)0 \neq Sx$ in place of Schema 1). However, Gödel's Proof could run in Robinson Arithmetic and all stronger arithmetics. It is thus incomplete.

This is fascinating, because this means that theories that could express only instances of arithmetic truth could be complete but any system that seeks to express generalizations or endeavor to uncover the laws of arithmetic truth would be inevitably incomplete.

Next, we shall first give my definition of a purely invented system and any system that does not fall within this definition will be at least a partially discovered system.

3.2 Definition of a Purely Invented System

The most important question in this matter is: how do you extinguish invention from discovery? One could argue that everything we express is invented because they all result from human language and human understanding. When we collide two particles together to get a split second of a heavy element, are we inventing that element? Or discovering it? Probably discovering it. In some sense, invention has more "freedom" than discovery. With discovery, there seems to be a right or wrong, a criterion of success. Invention is arbitrary and its derivations are restricted by human concept and definition. However, though unlikely, we can still unknowingly invent something and later find out that it is discovered.

A purely invented formal system is a system where all the axioms are defined by, relies on and are subject to human intelligence or other invented systems except logic.

Inevitably, since it is a formal system, it will include logic, or classic propositional logic and the rules of inference to create truth-preserving proofs and theorems.

3.3 Could Invented Systems be Incomplete

Invented Systems could very well be incomplete. In fact, it is very likely that an invented system is incomplete simply because it's not strong enough. For example, take the invented system I1 with similar language as arithmetic but instead of + and × we have a single operator ◦ which is defined by the following four schemas:

Schema 7. $0 \neq S\zeta$

Schema 8. $S\zeta = S\xi \rightarrow \zeta = \xi$

Schema 9. $0 \circ \zeta = S\zeta$

Schema 10. $S\zeta \circ S\xi = \zeta \circ \xi$

Schema 11. $\zeta \neq 0 \rightarrow \zeta = S^n(0)$

where $S^n(0)$ denote operand S applied a random n times to 0. I1 's axioms are derived by substituting numerical instances into these Schemas. I1 's language is quantifier and connective free and contains the logical symbols $\neg, \rightarrow, (,), 0, =, S$.

I1 is similar to arithmetic but evidently a completely different system.

Theorem 3. I1 is incomplete.

Proof. $I1 \not\vdash (\zeta \circ 0 = S\zeta) \vee I1 \not\vdash (\zeta \circ 0 \neq S\zeta)$

We can construct two interpretations, both satisfying the axioms of I1 , but in one of them $\zeta \circ 0 = S\zeta$ would be true, while in the other it would be false. Consider the following interpretation of I1 in the language of arithmetic:

$0 \rightarrow 0,$

$S \rightarrow S,$

$\zeta \circ \xi = |\zeta - \xi| + 1$

It can be easily shown that this always satisfies the schemas of I1 within the domain of arithmetic. Therefore it is a valid interpretation of I1 . Within this interpretation, $\zeta \circ 0 = |\zeta - 0| + 1 = S\zeta$, so it is true.

Now consider this different interpretation of I1 in the language of arithmetic:

$0 \rightarrow 0$

$S \rightarrow S$

$$\zeta \circ \xi = \begin{cases} |\zeta - \xi| + 1 & \zeta \leq \xi \\ |\zeta - \xi| & \zeta > \xi \end{cases}$$

It is evident that Schema 7 & 8 still hold. Schema 9 holds because 0 is always less or equal to ξ within the domain of the natural numbers. Schema 10 holds in both cases. So this is also a valid interpretation of I1. However, in this interpretation, $\zeta \circ 0 = S\zeta$ does not hold for any element in the domain for ζ except 0. Therefore, it is false.

Since there exists an interpretation of I1 where $\zeta \circ 0 = S\zeta$ is true and also one where it is false, $\zeta \circ 0 = S\zeta$ is unprovable in I1 .

To solve this problem, we could add another schema to system I1 to form I2 .

Schema 12. $\zeta \circ 0 = S\zeta$

Theorem 4. I2 is complete.

Proof. Since Schema 11 confines all elements in I2 to essentially whole numbers, this additional schema would allow us to evaluate all ◦ operations to a single number and since I2 is quantifier free and could only contain instance equations and parentheses, we could eventually simplify every parentheses in order and eventually both sides of an equation to a single number and see if they are the same to evaluate the equation's truth value.

As can be seen by this system, invented systems could be incomplete, but as we add unprovable but true claims to the set of axioms to build a slightly stronger system, completeness could sometimes be achieved. However, the problem with arithmetic and Gödel's proof is that no matter how many axioms you

try to add to an arithmetic-equipped system, it will always be incomplete. Invented systems could be incomplete, but they are not guaranteed incompleteness in the same way arithmetic is guaranteed essential incompleteness in which any consistent axiomatizable extension of it is also incomplete.

More broadly, we can imagine that we have a purely invented system I with purely arbitrary axioms that is not capable of arithmetic. Is I complete? If we have a false statement S in I , we could simply enumerate the theorems of I through the rules of inference until we come across S or $\neg S$. Hypothetically, even if we go on for billions of billions of theorems and still don't come across S or $\neg S$, I wouldn't be provably incomplete in the same manner as arithmetic. We can also just add S or $\neg S$ to the set of axioms if they preserve consistency to make a stronger, "more complete" system.

3.4 Why Essential Incompleteness Indicates Discovery

Imagine you are a sculptor, inventing and creating miniature statues out of mud. You would expect that you could control everything about your creations. You can move their arms, move their legs, anything you want. If you discover a hole in a statue's body, you can just use some additional mud to patch it up. However, what if when you craft it to resemble one specific god, you realize that you can never create a "complete" sculpture? When you patch up one hole on its leg, another appears on its arm? When you patch up that hole, another is discovered on its stomach. No matter how hard you try to fix your effigy, there always seems to be at least a hole appearing somewhere on its body. There is nothing you could do that could fix that figurine and you discover that this hole, this horrific and perpetual imperfection, is conceptually fundamental to the effigy itself. It doesn't affect any other shape you create out of the same mud, but only those with the likeness of this specific deity. Aside from being crept out by Lovecraftian strangeness, would you admit that you are discovering, not inventing, this strange phenomenon? Math is that god whose resemblance we are trying to capture with our mud, our language and logical system.

The inventor indicates some degree of control over the invented that is contradicted by the untameableness of arithmetic. Gödel's theorems did not just prove the incompleteness of arithmetic, but any extension of arithmetic. This unique property marks mathematics apart from other existing systems. But that raises the question, is arithmetic really unique in being essentially incomplete? What if we can invent completely original systems that are also essentially incomplete? That would threaten the entire argument.

3.5 Essentially Incomplete Invented Systems

So could invented systems be essentially incomplete? It is possible. The obvious method is to construct a proof similar to Gödel's that could consistently con-

struct an unprovable but true statement in a system that could be expanded to any extensions of that system. The best example of such a proof would be the halting problem which uses the diagonalization method like Gödel. The Turing Machine, though it is not a strict logical theory, exhibits a similar phenomenon as mathematics because a Turing Machine cannot figure whether any given Turing Machine would halt.

So what if we try to apply Gödel's method to any invented system? We would want to construct a true but unprovable sentence in that system using the fundamental elements of that system. There must be a few conditions to be met for Gödel's method to work.

1. There must be a "Gödel Numbering" function. There must be a mechanical way to inject all valid statements onto the constant domain of IG .

2. There must be a theorem similar to the Diagonalization Lemma. The system must be capable of processing functions about its elements that indicate statements about the sentence those elements correspond to and specifically its own corresponding element.

In these theorems, constants are meant to be parallel to arithmetic's numbers, as in terms either originally an element (0 in mathematics) or formed through applications of unary functions (such as S) of those original elements (S0, SS0, etc). From the first condition, we know that all invented systems with a limited number of generated constants cannot be proven incomplete using Gödel's method because in any logical theory, the number of logical sentences is infinite and cannot form an injection onto a finite set of constants.

So what about systems with infinite constant elements? Since this is an invented system, it would make sense if the elements are Turing enumerable. But first, we shall define two generalized versions of Gödel Numbering and the Diagonalization Lemma necessary for Gödel's proof:

Assumption 1 (Generalized Gödel Numbering). In a logical theory I with any language, if there exists a Turing computable method to map any sentence in I to a constant operand of I, then we call the operand that sentence S maps to using this method the "Gödel Expression" of S, denoted by $\ulcorner S \urcorner$.

Assumption 2 (Generalized Diagonalization Lemma). In a logical theory I with any language where Gödel Expression exists, if for every valid unary formula $\phi(x)$ definable in the language of I where x is an operand, there exists a sentence S in the language of I such that: $S \leftrightarrow \psi(\ulcorner S \urcorner)$, then system I satisfies the Generalized Diagonalization Lemma.

Since we know from earlier that unprovableness is definable in all axiomatizable systems and our invented system is no doubt one, we can know that if there exists in theory I both Gödel Expression and the Generalized Diagonalization Lemma is true in I, then I can be proven to be incomplete using Gödel's method, constructing a Gödel sentence within the language of I.

3.6 Essentially Incomplete Invented Systems Can Express Arithmetic

As mentioned earlier, though the Turing Machine is not a logical theory, it exhibits a similar property to incompleteness in the unsolvability of the halting problem. However, another property of the Turing Machine is that it can express and calculate arithmetic and we wonder if all provably incomplete systems can do the same.

Conjecture 1. If a purely invented logical theory has Gödel Expression and satisfies the Generalized Diagonalization Lemma, then it could express arithmetical relationships.

Let's assume that IG can be proven to be incomplete using Gödel's method. That would mean that there exists within IG a Turing computable function to map every sentence in IG to an element in IG. We also know that if we use Gödel's mapping process, there is also a unique Gödel number to every sentence. The set of all sentences in IG should form an injection onto both the set of natural numbers and the set of all elements of IG. Through this bridge, we can form a bijection between the natural numbers to the elements of IG.

Through this method, we have found a way to define specific numbers in IG. If we can find a function definable in IG that could define addition and multiplication between these "numbers," we can then prove Conjecture 1. This means we would have a function that would take in two constants in IG, one whose corresponding Gödel expression in Assumption 1 mapped through Gödel numbering is 2 and another mapped to 3. That function would then produce a constant whose Gödel expression mapped through Gödel numbering is 5 and it would be true for all combinations of addition, same for multiplication. We have been unable to find such a function as of right now. The main obstacle would be to utilize Assumption 2 in some way to aid in that endeavor because Assumption 2 is a strong statement whereas Assumption 1 is the weaker one. We have thought of methods such as creating a slightly different system for Gödel numbering that could make addition or multiplication easier to be expressed in the language but there hasn't been any fruition in that direction currently.

3.7 Implications of Conjecture 1

If we have successfully proved Conjecture 1, what does it signify? It would mean that arithmetic is now not only a sufficient condition for essential incompleteness, but also a necessary condition, granting it truly unique and fundamental status in our logical systems. We would think we are still inventing the systems and the axioms themselves, but what we are actually doing is inventing another language, another medium for arithmetical laws and properties to manifest (Tall, D). It would mean that any essentially incomplete systems that utilize

logic and human language are in essence just another interpretation of arithmetic with extra steps. Math's unique imperfection would seem fundamental to the unfathomability of the universe itself and thus prove that under our current definitions, it is most definitely discovered rather than invented.

4. Responding to Some Arguments Against the Discovery of Math

A common argument cites the triviality of some mathematical concepts such as the Erdos number and language as evidence for the invention of math. This argument does not go contrary to the point we are making. Though the concept of arithmetic is discovered, we could still have invented some mathematical concepts and the mathematical language, but we invented them on the basis of and within a discovered arithmetic. We discovered metal but we invented swords and spears. Therefore, we ought to separate the language of math from the abstract concept of math itself. The former is invented whereas the latter discovered.

For arguments on the "fickleness" of mathematics, it is true that our definitions of arithmetic have changed throughout the centuries but that only means that our way of interpreting math has changed and become more accurate. As we have stated, the language and axioms that we use to define the system of arithmetic are not math itself but invented interpretations and what Gödel proved was that we can never invent an interpretation that can fully capture the concept and system of math. The system itself is discovered but what for and how we choose to use the system is invented.

As for arguments on a different "alien" mathematics or lack of mathematical concepts, they will definitely have a completely different system or even none at all, but no matter what system or language they come up with, we know that they also cannot also "prove" every mathematical "truth," whatever their counterparts are. If they are similar rational beings capable of logical reasoning and creating definitions for systems, then incompleteness still stands. Conjecture 1 in some ways solves that problem completely as it broadens the definition of "arithmetical systems" to any essentially incomplete systems. We trust that aliens would be just as exasperated with math as we are. At the end of the day, we are all mortals chained to not the essence, but the manifestations of the universe.

5. Conclusion

In this paper, we have summarized Gödel's proof of the essential incompleteness of arithmetic, in order to aid the understanding of why math is discovered rather than invented because of its unique essential incompleteness in contrast

to other purely invented systems. We have also conjectured that the ability to express arithmetic is not only a sufficient condition to essential incompleteness in a logical theory, but also a necessary one. Finally, under our new model, we have answered some common arguments for math is invented. For the future study of this topic, we would hope to prove or disprove our conjecture that any system that could be proved essentially incomplete using a similar method to Gödel would be capable of a certain degree of arithmetic.

When I first divined this connection between Gödel and Math's discoveredness, I saw in my mind a metaphor for the relationship this revelation revealed. Math is an abstract painting hidden behind the dark wall of concrete reality. Our ancestors poked tiny holes in the wall to gaze upon fractions of the painting from different angles. Since we cannot reach the painting, we used our minds to create our own "models" of the painting of math based on our observations through our peepholes.

These models are the systems of math that we have constructed over the ages. As time goes on, our collective intelligence enlarges old holes and pokes new holes. We do invent. The holes that we poke and the models that we build are our inventions, attempts to understand this painting further. Through more knowledge of the painting, we are able to construct newer models that can more accurately describe the painting. What Gödel proved was that we can never replicate the painting of math behind the dark wall of reality. We can never bring down the dark wall of reality and expose the painting of math to light entirely. This can only mean that this painting does not belong merely in the material world. If it is something we invented, then we ought to at least know that it could be understood, not know that we can never reach it. There will always be an impenetrable distance between us and mathematics, but that is precisely the reason why it is a discovered wonder of the universe.

References

- [1] Barker, A. (1994). Ptolemy's Pythagoreans, Archytas, and Plato's conception of mathematics. *Phronesis*, 39(2), 113-135.
- [2] Burkholder, L. (1987). The halting problem. *ACM SIGACT News*, 18(3), 48-60.
- [3] Burnyeat, M. (2000). Plato on why mathematics is good for the soul.
- [4] Closer To Truth. (2020, April 13). Roger Penrose - Is Mathematics Invented or Discovered? [Video]. YouTube. <https://youtu.be/ujvS2K06dg4?si=ILKTQbubJfpjKk1b>
- [5] Davey, K. (2023). *Advanced Logic Notes* [PDF].
- [6] K. Davey, personal communication, 2023
- [7] Dehaene, S. (2011). *The number sense: How the mind creates mathematics*. Oxford University press.
- [8] Fine, K. (2012). Mathematics: discovery or invention?. *Think*, 11(32), 11-27]
- [9] Gödel, K., & Feferman, S. (1990). *Collected works*. Oxford University Press.
- [10] Goffman, C. (1969). And What Is Your Erdos Number? *The American Mathematical Monthly*, 76(7), 791-791. <https://doi.org/10.2307/2317868>
- [11] A. Guerrini, personal communication, June 10, 2023
- [12] D. Kremm, personal communication, December, 2022
- [13] Penrose, R. (2016). *The emperor's new mind: Concerning computers, Minds, and the laws of physics* [PDF]. Oxford University Press.
- [14] Prokopenko, M., Harré, M., Lizier, J., Boschetti, F., Peppas, P., & Kauffman, S. (2019). Self-referential basis of undecidable dynamics: From the Liar paradox and the halting problem to the edge of chaos. *Physics of life reviews*, 31, 134-156.
- [15] Sipser, M. (1996). *Introduction to the Theory of Computation*. *ACM Sigact News*, 27(1), 27-29.
- [16] Smith, P. (2021). Gödel without (too many) tears [PDF]. *Logic Matters*.
- [17] Tall, D. (2011). Crystalline Concepts In Long-term Mathematical Invention And Discovery. *For the Learning of Mathematics*, 31(1), 3-8. <http://www.jstor.org/stable/41319544>
- [18] TED-Ed. (2014, October 27). Is math discovered or invented? - Jeff Dekofsky [Video]. YouTube. <https://youtu.be/XxR5Kes4Rs>
- [19] Wang, H. (1997). *A logical journey: From Gödel to philosophy*. MIT press.
- [20] Wigner, E. P. (1995). The unreasonable effectiveness of mathematics in the natural sciences. In *Philosophical Reflections and Syntheses* (pp. 534- 549). Berlin, Heidelberg: Springer Berlin Heidelberg.