Comparison of Markowitz Model and Index Model in Optimization of Portfolio

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Abstract. To compare the differences between Markowitz Model and the Single Index Model, we have used historical return data for ten stock which belong in groups to three different equity sectors to practically implement the Markowitz Model and the Single Index Model. We find the stocks in the same equity sector being noticeably positively correlated even after eliminating systemic risks, which violates the formal condition of the IM. Also, as a result of very detailed comparison, we can conclude that the IM serves as an accurate approximation of the MM in practical applications for large enough number of risky assets. And adding a broad index such as S&P 500 to an existing individual equities portfolio is improving its properties.

Keywords: Markowitz Model, Single Index Model, Sharpe Ratio.

1. Introduction

Since the 1980s, people have focused on stocks and securities in developed and emerging markets. In the last 30 years, there have been wider choices of assets and asset classes available for use in asset allocation. From the perspective of expected return, the stock market has great attraction, but they also have considerable risks. From the Latin American debt crisis in the 1980s to the Technology Bubble in 2000 and the Subprime Lending Crisis of 2008, these examples illustrate how destructive of the investor capital the equity markets can be. To avoid repeating history, we should learn historical lessons profoundly. This is why people are paying more and more attention to investment risk management and investment returns.

Portfolio theory is identified as the quantitative analysis of optimal risk management. As early as 1952, Markowitz [1] introduced the mean-variance theory, which pioneered using quantitative ideas to construct portfolios. Since then, there have been other scholars who have further investigated the Markowitz model. For example, in 1979 Love [3] attempted to develop a model based on the Markowitz model that would allow one to study the effect of diversification on export losses. In 1993 Gollinger et al. [4] made the first attempt to calculate the efficient frontier of a commercial loan portfolio based on the structure of the Markowitz equity portfolio model. More recently, in 2020, Shadabfar et al. [5] used a probabilistic approach to optimal portfolio selection using a mixture of Monte Carlo simulation and the Markowitz Model.

To further refine the Markowitz Model, in 1963 William Shape proposed the Single Index Model [2], which significantly promoted the practical application of portfolio theory. Other scholars have further investigated it since then. For example, in 1986 Collins et al. [6] used Single Index Model for risk analysis in farm planning applications. In 2010 Galea et al. [7] studied structural Sharpe models under t-distribution. In 2018 Mallikharjunarao et al. [8] constructed optimal portfolios in two sectors: IT and Pharma, utilizing Sharpe index models.

On this basis, other scholars have compared these two models. For instance, in 1989 Seler [9] compared the Markowitz Mean-Variance Model and Sharpe Single Index Model to construct the Istanbul Stock Exchange portfolios for the period 1986-1987. In 2011 Bekhet et al. [10] compared the Markowitz and Single Index Models to construct portfolios of Amman Stock Exchange (ASE) companies. In 2021 Susanti et al. [11] compare the best portfolio formation results of the Markowitz and single index models for LQ index 45 in the COVID-19 pandemic.

To make a better comparison of the Markowitz Model and the single index model, we select ten stocks from three industries over the past 20 years as the testing sample of stocks, and the Markowitz Model and the Index Model are used to construct portfolios with different constraints. We compared the differences between the two models by calculating two crucial points on the Efficient Frontier: the Minimal Risk portfolio and the Efficient Risky Portfolio (or Maximum Sharpe Portfolio). Furthermore, we tested whether a broad equity index, which is a diversified stock portfolio, added to the model will make a difference between two models.

We organized the rest of the article as follows. An introduction to the theory of risky portfolios models is provided in Section 2. In Section 3, we preprocess the data and calculate the matrix of correlation coefficients between stocks. In Sections 4 analyze the MM and IM model results. Conclusions and future research directions are presented in Section 5.

2. The MM Model and The IS Model

The Markowitz Model (MM) is based on the following assumptions:

1. The investor considers each possible investment choice based on the distribution of assets return over the time of a given position.
2. The investor estimates the risk of a portfolio based on the variance or standard deviation of the expected return of the assets.
3. The investor's decision is solely based on the risk and expected return of the assets and securities.
4. For a given level of risk, the investor prefers to maximize the expected return; or for a given level of expected return, the investor prefers to minimize the risk.

The expected return for the Markowitz Model (MM) is:

$$R_p = \sum_{i=1}^{n} w_i \, r_i.$$

$$\sigma_p = \sqrt{\sum_{i=1}^{n} \ \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j)}.$$

where we denoted:

r_i, r_j: The expected return of the assets i and j;

w_i, w_j: The proportions of assets i and j in the portfolio;

i and j: The indices enumerating the two assets;

$Cov(r_i, r_j)$: The covariance between the two assets, which measures the strength of correlation between two assets.

William Sharpe's Single-Index Model is based on the following assumptions:

1. The risk of the portfolio is divided into systematic risk and nonsystematic or idiosyncratic risk. The external factor (the index) will not affect the nonsystematic risk.

2. The idiosyncratic risk of one asset will not affect the idiosyncratic risk of another asset. The correlation of two assets depends solely on the joint response of the external factors:

The expected return for the Single-Index model is:

$R_p = \sum_{i=1}^{n} w_i \, r_i$.

The standard deviation for the Single-Index model is:

$\sigma_p = \sqrt{\sum_{i=1}^{n} (w_i \beta_i)^2 \, {\sigma_M}^2 + \sum_{i=1}^{n} {w_i}^2 \, {\sigma_i}^2}$.

where we denoted:

r_i, r_j: The expected returns of the assets i and j;

w_i: The proportion that asset i occupies in our portfolio;

n: The number of assets;

β_i: The risk factor of asset i;

σ_M: The systematic risk;

σ_i: The nonsystematic risk.

Theoretically, IM model can be regarded as a simplified version of MM model. The MM model requires to calculate the correlation coefficient between every two assets. However, in the actual process of portfolio construction, there are too many assets to choose, which makes the calculation of the covariance matrix of the portfolio too many parameters to be calculated, which brings more inconvenience in the actual use process. The IM model simplifies the way of calculating the covariance matrix of the MM model, and converts the covariance between two assets into the specific risk of each asset and the exposure to systemic risk, which greatly reduces the number of parameters that the model needs to calculate when constructing the optimal portfolio.

3. Data and descriptive statistics

Even with the use of IM theory, there are too many assets in the market that can be used to build portfolios. In order to make our analysis process as simple as possible without losing representativeness, we have selected several representative companies from the three industries of technology, financial services and industry as the assets used to build portfolios. Companies from the financial Services include: Bank of America Corporation(BAC), Citigroup Inc.(C), Wells Fargo & Company(WFC), The Travelers Companies, Inc.(TRV). Companies from the Industry include: Southwest Airlines(LUV), Alaska Air Group, Inc.(ALK), Hawaiian Holdings, Inc.(HA). Companies from the technology industry include: Adobe Inc.(ADBE), International Business Machines Corporation(IBM), SAP SE(SAP). In order to reduce the impact of the non-normal distribution of specific risk on the IM model, we use the monthly yield data for analysis. The range of dates for which we have gathered is from 05/2001 to 5/2021, which approximately corresponds to the previous 20 years of historical data. The risk-free interest rate is assumed to be 1.30%, that is, the average yield of three-month US treasury over the past 20 years The basic information about the stocks used is shown in the following table:

Table 1: descriptive statistics of row return

	BAC	C	WFC	TRV	LUV	ALK	HA	ADBE	IBM	SAP
Average Return	0.83 %	0.04 %	0.62 %	0.69 %	0.92 %	1.56 %	2.42 %	1.73 %	0.30 %	0.99 %
Standard Deviation	11.35 %	12.28 %	8.15 %	5.78 %	9.15 %	10.87 %	17.90 %	9.19 %	6.72 %	9.81 %
Median	0.56 %	0.57 %	0.84 %	1.27 %	1.14 %	1.68 %	1.60 %	2.77 %	0.33 %	0.66 %
Max	72.66 %	68.67 %	40.52 %	19.74 %	32.29 %	34.52 %	99.28 %	28.08 %	35.38 %	70.13 %
Min	53.27 %	57.75 %	35.98 %	19.81 %	26.57 %	43.58 %	50.00 %	32.51 %	23.66 %	41.56 %

In order to better understand the correlation between the returns of these companies, we calculated the correlation coefficient matrix between the companies. From the correlation coefficient matrix, we can see that the returns of companies in the same industry are highly correlated, which is particularly prominent in the financial services industry. At the same time, in order to better analyze the relationship between market returns and individual stock returns, we added market returns represented by SPX to the correlation coefficient matrix. We find that the financial services industry and the technology industry have a higher correlation with market returns than industry, which in a sense reflects that the financial services industry and the technology industry have a higher weight than industry in the economic structure of the United States. For better visualization, we have exhibited the data as both numerical and as a heatmap coloring the numerical data in the Table2 below. The darker color of a table cell signifies the larger correlation coefficient between the two assets. The lighter color of a table cell signifies the weaker correlation coefficient between the two assets.

Table 2: Correlation coefficient matrix for row return

	BAC	C	WFC	TRV	LUV	ALK	HA	ADBE	IBM	SAP	SPX
BAC	1.000										
C	0.824	1.000									
WFC	0.760	0.701	1.000								
TRV	0.388	0.511	0.340	1.000							
LUV	0.428	0.427	0.401	0.406	1.000						
ALK	0.279	0.302	0.342	0.361	0.517	1.000					
HA	0.334	0.342	0.354	0.240	0.422	0.402	1.000				
ADBE	0.424	0.464	0.293	0.437	0.379	0.226	0.174	1.000			
IBM	0.312	0.409	0.264	0.373	0.337	0.347	0.240	0.450	1.000		
SAP	0.331	0.429	0.297	0.366	0.313	0.282	0.142	0.537	0.586	1.000	
SPX	0.601	0.695	0.548	0.594	0.531	0.460	0.385	0.654	0.638	0.643	1.000

The IS model assumes that the risks in the company can be divided into systematic risks and idiosyncratic risks, among which the idiosyncratic risks of different companies should be uncorrelated. In order to extract the idiosyncratic risk of each company, we use the data of the past 20 years for regression to get the exposure of each company to systemic risk β, and then we calculate the idiosyncratic risk of each company. The results of regression are shown in Table 3.

Table 3: Regression results of each company

	BAC	C	WFC	TRV	LUV	ALK	HA	ADBE	IBM	SAP	SPX
β	1.60	2.01	1.05	0.80	1.15	1.18	1.63	1.42	1.01	1.48	1.00
α	-0.01	-0.14	0.01	0.03	0.01	0.09	0.15	0.09	-0.03	0.01	0.00
Residual Stdev	31.4%	30.3%	23.4%	16.0%	26.8%	33.4%	57.2%	23.8%	17.6%	25.8%	0.0%

From Table 3, we can see that C (Citibank) has the highest exposure to systemic risk, while HA (Hawaiian Airlines) has the highest idiosyncratic risk. From the perspective of business results, HA (Hawaiian Airlines) has the highest α， And C (Citibank) has the smallest α.

We are more concerned about whether there is correlation between the regression residuals of each company. For this reason, we calculated the correlation matrix of the residuals of each company, and the results are shown in Table 4.

Table 4: Correlation coefficient matrix for residuals

	BAC	C	WFC	TRV	LUV	ALK	HA	ADBE	IBM	SAP	SPX
BAC	1.000										
C	0.707	1.000									
WFC	0.644	0.532	1.000								
TRV	0.049	0.170	0.021	1.000							
LUV	0.160	0.094	0.154	0.132	1.000						
ALK	0.001	-0.031	0.120	0.121	0.360	1.000					
HA	0.138	0.112	0.184	0.014	0.277	0.274	1.000				
ADBE	0.052	0.018	-0.103	0.081	0.048	-0.113	-0.111	1.000			
IBM	-0.115	-0.059	-0.131	-0.007	-0.002	0.077	-0.008	0.058	1.000		
SAP	-0.090	-0.032	-0.087	-0.026	-0.045	-0.023	-0.150	0.202	0.299	1.000	
SPY	0.019	0.032	-0.021	-0.034	-0.115	-0.022	-0.014	-0.049	0.052	-0.003	1.000

From Table 4, we can see that there is basically no correlation between the residual items among different industries, but within the financial services industry, there is still a high correlation between the residual items among companies, which violates the premise of the IS model and, in a sense, implies the existence of other industry-related risks for scholars.

4. The Comparison of MM and IM model

We compared the differences between the two models by calculating two crucial points on the Efficient Frontier: the Minimal Risk portfolio and the Efficient Risky Portfolio (or Maximum Sharpe Portfolio).

First of all, we consider a free problem. That is, when the index can be invested and all assets can be short indefinitely, what is the difference between the minimum risk portfolio and the efficient risk portfolio calculated by the MM model and the IS model. Figure 1 and Figure 2 show the proportion of each asset in the minimum risk portfolio and the optimal risk portfolio calculated using MM model and IS model respectively.

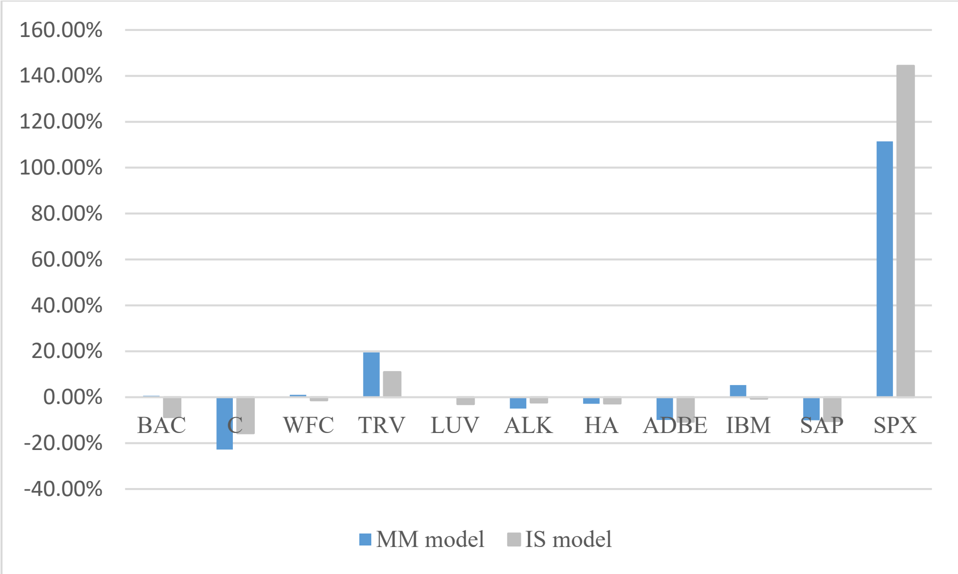

Figure 1: Proportion of assets in the minimum risk portfolio under free problem

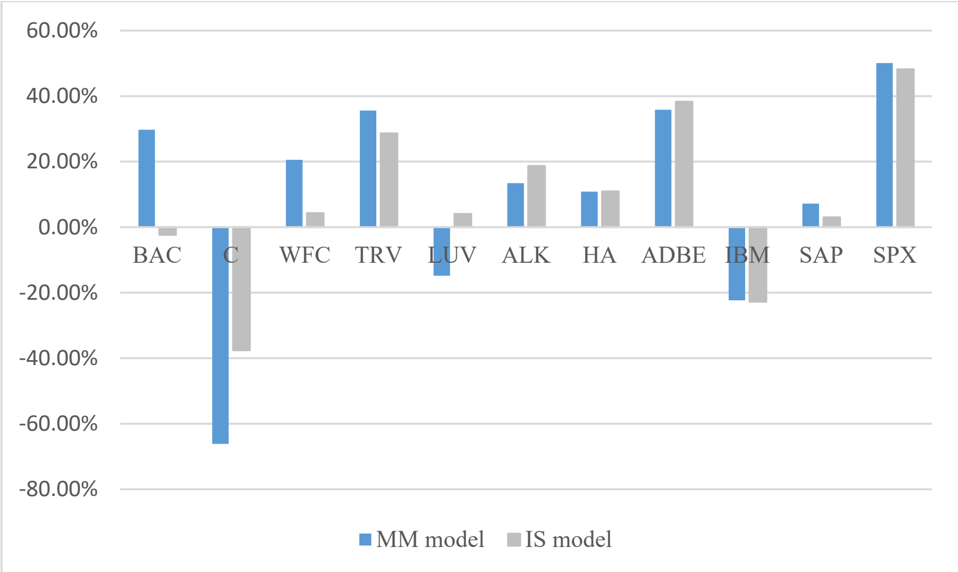

Figure 2: Proportion of assets in the optimal risk portfolio under free problem

From Figure 1 and Figure 2, it can be seen that the proportion of assets calculated using different models has a certain gap in value, but the direction is basically the same. In order to better see the difference between the two models, we will further observe the income, standard version and Sharp ratio of the portfolio. This result is shown in Table 5.

Table 5: the results of the problem under free problem

	Model	Return	Stdev	Sharpe Ratio
Min VAR	MM model	6.72%	11.75%	0.461
	IS model	5.85%	11.95%	0.381
Max Sharpe	MM model	22.07%	21.33%	0.974
	IS model	19.81%	21.99%	0.842

From the results in Table 5, we can see that the minimum risk portfolio and optimal risk portfolio constructed by MM model has higher returns and smaller variance than IS model. But the difference between the calculation results of the two models is not very big.

To better simulate the Regulation T by FINRA, which allows broker-dealers to allow their customers to have positions, 50% or more of which are funded by the customer's account equity, we add additional optimization constraint is: $\sum_{i=1}^{11}|w_i| \leq 2$. Figure 3 and Figure 4 show the proportion of each asset in the minimum risk portfolio and the optimal risk portfolio

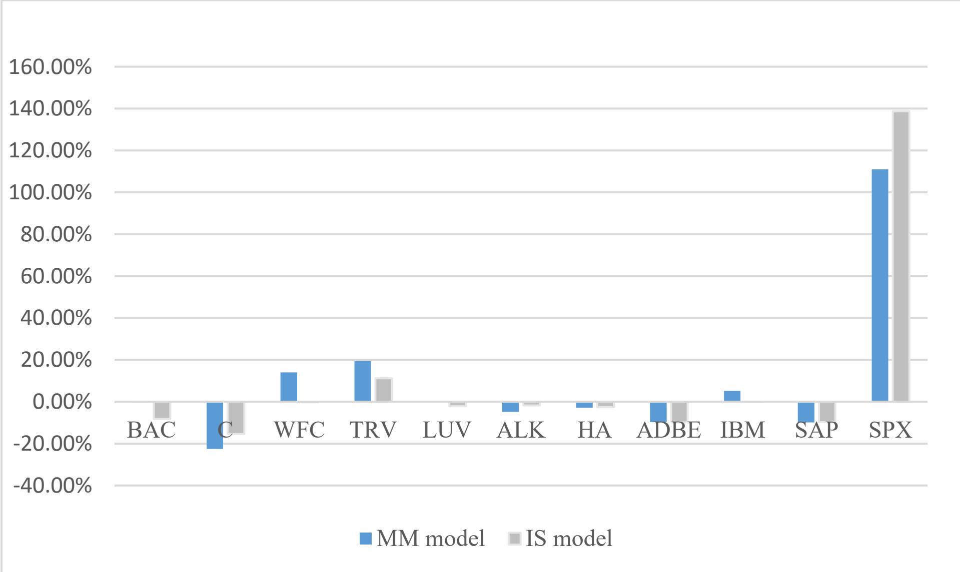

Figure 3: Proportion of assets in the minimum risk portfolio under Regulation T

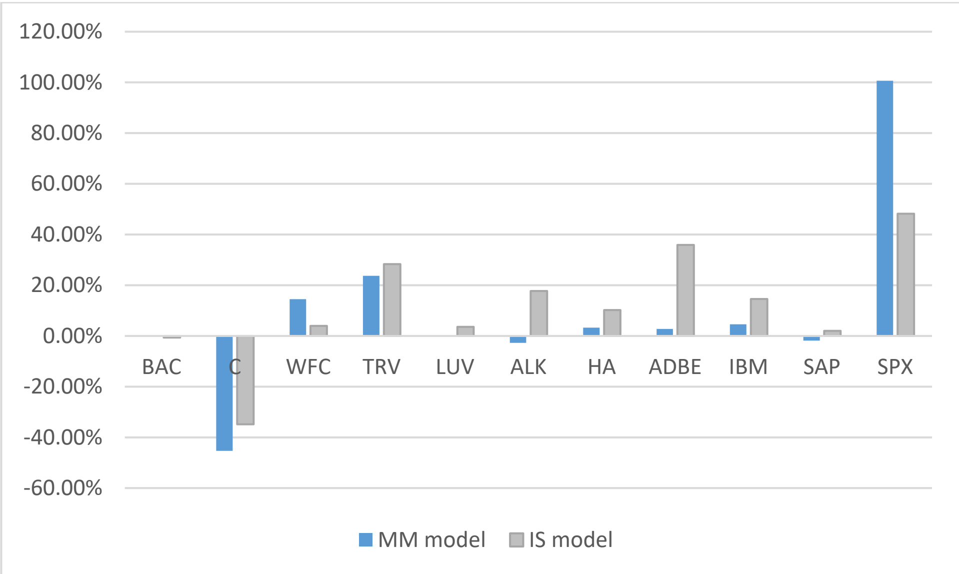

Figure 4: Proportion of assets in the optimal risk portfolio under Regulation T

From the results of Figure 3 and Figure 4, after adding regulation T as the most restrictive condition, the calculated proportion of each asset has a certain change compared with unrestricted, but the direction remains basically unchanged.

Table 6: the results of the problem under Regulation T

	Model	Return	Stdev	Sharpe Ratio
Min VAR	MM model	6.69%	11.75%	0.459
	IS model	6.07%	11.96%	0.399
Max Sharpe	MM model	11.55%	14.40%	0.712
	IS model	18.90%	21.04%	0.837

From the results in Table 6, we can see that the minimum risk portfolio constructed by MM model has higher returns and smaller variance when constructing the minimum risk portfolio. When constructing the optimal portfolio, the optimal risk portfolio constructed by IS model has a higher return and Sharp ratio. Compared with the case without constraints, the Sharpe ratio of the minimum risk portfolio and the optimal risk portfolio decreased after adding constraints.

From the previous two cases, SPX has a large proportion in the portfolio. Lastly, we would like to see if the exclusion of the broad index into our portfolio has positive or negative effect. The results are shown in Table 7.

Table 7: the results when exclude SPX

	Model	Return	Stdev	Sharpe Ratio
Min VAR	MM model	9.38%	15.45%	0.523
	IS model	9.26%	16.64%	0.478
Max Sharpe	MM model	26.53%	25.98%	0.971
	IS model	23.79%	26.68%	0.843

From the table 7, we can see that when calculating the minimum risk portfolio, the standard deviation of the minimum risk portfolio is less than 12% when there are no constraints or short constraints, and when SPX is excluded, even when there are no other constraints, the standard deviation of the minimum risk portfolio calculated by the two models exceeds 15%. When calculating the optimal risk portfolio, when SPX is excluded, the maximum Sharp ratio is very close to that in the free problem.

From this conclusion, we can see that adding SPX as an investable asset has a strong positive effect in building the minimum risk portfolio, but when building the optimal risk portfolio, without considering the transaction friction, we can build a portfolio similar to the index through diversified asset portfolios, so adding SPX can also have a certain positive effect in building the optimal risk portfolio, But the effect is not very significant.

5. Conclusion

To make a better comparison of the Markowitz Model and the single index model, we select ten stocks from three industries over the past 20 years as the testing sample of stocks, and the Markowitz Model and the Index Model are used to construct portfolios with different constraints. In this work we have arrived to the following conclusions. First, the stocks in the same equity sector being noticeably positively correlated. Even after eliminating systemic risks, there is still a high correlation between enterprises in the financial services industry, which violates the formal condition of the IM model. in a sense, it implies the existence of other industry-related risks. Second, as a result of very detailed comparison between the MM and IM models, we can convincingly conclude that the IM serves as an accurate approximation of the MM in practical applications for large enough number of risky assets. Lastly, we find that adding a broad index such as S&P 500 to an existing individual equities portfolio is improving its properties. Our investigation provides a lot of numerical evidence to support these conclusions. Still, We find that MM model can calculate the effective frontier more accurately than IM model when the number of assets is small, but as the number of assets available increases, the difficulty of parameter estimation will make IS model more practical.

However, our investigation still has some limitations. One of the limitations is that we have only used the data for ten stocks, which is very limited. There are thousands of assets in the market and the data of ten stocks is not diverse enough for us to be certain that the portfolio of other stocks, funds, or bonds will result in the same conclusions that we have drawn with 10 stocks. Additionally, the Markowitz Model needs the data on expected return and the expected standard deviation. In our study we have instead estimated then from sample historical data. Calculation of the actual expected values will be extremely hard. Also, the current investigation only compares the IM and MM results under the free problem and short-sales constraints, the results in other constraints are not investigated.

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